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Mathematical Reviews

Vol. 19, No. 1 JANUARY 1958 pp. 1-110

FOUNDATIONS, THEORY OF SETS, LOGIC

Le Page, Th. H. Symétrie et structures mathématiques. Acad. Roy. Belg. Bull. Cl. Sci. (5) 42 (1956), 1241–1249.

A popular exposition, summed up by the author's words: "Chaque fois que l'on se trouve en présence d'un ensemble d'éléments, pour lequel on a reconnu certain type de structure, le problème se pose de déterminer son groupe d'automorphismes."

Grosheide F. W. zn, G. H. A. The scientific foundations of elementary mathematics. Axiomatics and geometry. Euclides, Groningen 32 (1956/57), 257–277. (Dutch)

A lecture delivered during the "vacation-course 1956" of the Mathematisch Centrum.

★ **Wittgenstein, Ludwig. Remarks on the foundations of mathematics.** Edited by G. H. von Wright; R. Rhees; G. E. M. Anscombe. Translated by G. E. M. Anscombe. [The original German text is given on the left-hand pages.] The Macmillan Co., New York, 1956. xix+xix+196+196+197–204 pp. $5.75.

This is a collection of previously unpublished manuscripts. The mode of presentation (juxtaposition of German and English) is the same as in "Tractatus logico-philosophicus" [Paul-Trench-Trubner, London, 1922], and a word of praise is due to the translator for coping successfully with a difficult task.

This volume is not a contribution to the technical problem of laying the foundations of a branch of mathematics or of mathematics as a whole. Rather, it is characterized by a number of searching questions which go through the entire volume and which are beyond a purely formal discussion of the subject. What do we really mean by an inference or a proof? What do we mean by "applying a rule correctly"? How do we come to understand and accept somebody else's arguments? These questions are raised again and again in numerous forms and shapes. Those who are willing to take time off from their bread-and-butter mathematical activities to reflect on such questions will find this book very stimulating. On the other hand, those mathematicians who have the professional weakness of expecting precise results and conclusive answers to their problems will be disappointed. Consider, for example, Gödel's incompleteness theorem. On p. 174, the author states, "My task is not to talk about (e.g.) Gödel's theorem but to talk around it". [The translation has the misleading phrase "··· but to pass it by".] This is done on pp. 50–54 and 176–177. A reader who is familiar with Gödel's work may well feel that he has still not appreciated the full philosophical depth and consequence of the theorem. He will follow Wittgenstein's discussion with great interest. But he will be disappointed to find that the concluding remark is nothing more than — "However queer it sounds, my task as far as concerns Gödel's proof seems merely to consist in making clear what such a proposition as: 'suppose this could be proved' means in mathematics.

To a mathematician, this book is not a Pegasus but a gadfly. But, as somebody once remarked, such may be the main purpose of a philosopher. *A. Robinson.*

★ **Boole, George. An investigation of the laws of thought, on which are founded the mathematical theories of logic and probabilities.** Dover Publications, Inc., New York, 1957. xi+424 pp. (1 plate). $2.00.

A reprinting of the first American edition of 1854.

★ **Carroll, Lewis (C. L. Dodgson). Symbolic logic. I. Elementary.** 4th ed. Berkeley Enterprises, Inc., New York, N. Y., 1955. xxxi+199 pp. 2 s.

★ **Chauvineau, Jean. La logique moderne.** Presses Universitaires de France, Paris, 1957. 128 pp.

An exposition with four chapters: logique propositionelle, logique fonctionelle, logique propositionelle déductive, logique fonctionelle déductive.

★ **Goodstein, R. L. Mathematical logic.** Leicester University Press, 1957. viii+104 pp. 21 s.

In the words of the author, "The aim of this little book is to introduce teachers of mathematics to some of the remarkable results which have been obtained in mathematical logic during the past twenty-five years. The book is designed to be read by mathematicians who have little or no previous knowledge of symbolic logic, and is largely self-contained in the sense that the proofs of major results are given in detail. A great many different facets of the subject have been briefly sketched, but rigour has not been intentionally sacrificed for ease of reading, nor has generality been pursued for its own sake."

That a remarkable number of topics has been covered is clear from the following list of chapters and their (partial) contents: Introduction, The Function of Mathematical Logic; Chap. I, The Sentence Calculus (Truth tables, three-valued logic, axiomatic theory, intuitionistic logic, Łukasiewicz's bracket-free notation, natural inference); Chap. II, Predicate Calculus (axiomatic theory, deduction theorem, (decision method for monadic predicates, Gödel's theorem); Chap. III, Number Theory (recursive functions, λ-conversion, recursive arithmetic); Chap. IV, The Incompleteness of Arithmetic (Gödel numbering etc., undecidability of arithmetic and of predicate logic); Chap. V. Extended Predicate Logic (class logic, stratification, paradox of the class of subclasses); Notes and Bibliography; Index. That the treatment is addressed to the mature mathematician is evidenced by the provision of only essential details. Indeed, the mathematician unfamiliar with modern mathematical logic may find the going rough in spots, so much is condensed into a short space; a circumstance which is sometimes aggravated by involved sentences. An instance of this is the fourth paragraph on p. 50, which consists of one sentence of ten lines. Only twelve typo-

graphical or symbolic errors were noted by the reviewer; none that the careful reader cannot clear up for himself. The mathematician who wishes to know something of modern logic will certainly gain much by reading this little book. It could also be used in a short course or seminar for advanced students; although there are no formal exercises, many such suggest themselves.

R. L. Wilder (Ann Arbor, Mich.).

✱ **Клини, Стефен К. [Klini, Stefen K. (Kleene, S. C.)] Введение в метаматематику. [Introduction to metamathematics.]** Izdat. Inostr. Lit., Moscow, 1957. 526 pp. 32.25 rubles.

A translation by A. S. Esenin-Vol'pin under the editorship of V. A. Uspenskiĭ from the English of the book reviewed in MR **14**, 525.

Ladrière, Jean. La notion de constructivité en métamathématique. Bull. Soc. Math. Belg. **8** (1956), 82–97.

A rapid historical survey.

Pap, Arthur. Mathematics, abstract entities, and modern semantics. Sci. Monthly **85** (1957), 29–40.

An expository discussion of present-day mathematical aspects of the nominalist-realist dispute.

Shapiro, Norman. Degrees of computability. Trans. Amer. Math. Soc. **82** (1956), 281–299.

The Kleene hierarchy as usually described in the literature is a hierarchy of sets of non-negative integers; i.e., total (always defined) relations. The author in this paper extends the hierarchy to include partial relations, and finds the place of several particular relations in this extended hierarchy.

Part I defines the extended hierarchy and establishes the analogs of several well-known theorems for it. The reviewer found it remarkably obscure, and feels it worth while to restate the basic definitions and theorems in (to him) more meaningful terms. Definitions: A relation is an ordered pair $\langle\alpha, \beta\rangle$ of disjoint sets of non-negative integers; a relation is total if $\alpha=\beta'$; $\langle\alpha, \alpha'\rangle$ is r.e. or recursive according as α is; $\langle\alpha, \beta\rangle$ is partial recursive if α and β are r.e.; $\langle\alpha, \beta\rangle\subset\langle\gamma, \delta\rangle$ when $\alpha\subset\gamma$, $\beta\subset\delta$; $\langle\alpha, \beta\rangle$ is potentially recursive, r.e. or partial recursive when $\langle\alpha, \beta\rangle\subset\langle\gamma, \delta\rangle$ and $\langle\gamma, \delta\rangle$ is recursive, r.e. or partial recursive; E_0 is a transformation on relations: $E_0\langle\alpha, \beta\rangle=\langle\Sigma_x (y|\, J(x, y) \in \alpha), \Pi_x (y|\, J(x, y) \in \beta)\rangle$; $\langle\alpha, \beta\rangle\ll\langle\gamma, \delta\rangle$ when there exists a partial recursive function $f(x)$ defined on $\alpha\cup\beta$ such that $\langle f(\alpha), f(\beta)\rangle\subset\langle\gamma, \delta\rangle$; α is anti-r.e. when α' is r.e. The kth level generalizations in the Kleene hierarchy of recursive, r.e., and anti-r.e. are k-recursive, k-enumerable and anti k-enumerable. Theorems: (1) The class of potentially k-enumerable relations is the closure under E_0 of either of the classes potentially k-recursive or potentially k-enumerable. (2) If $\langle\alpha, \beta\rangle\ll\langle\gamma, \delta\rangle$ and $\langle\gamma, \delta\rangle$ is potentially k-enumerable, then $\langle\alpha, \beta\rangle$ is. (3) $\langle\alpha, \beta\rangle$ is potentially k-partial recursive if and only if both $\langle\alpha, \beta\rangle$ and $\langle\beta, \alpha\rangle$ are potentially k-enumerable.

Parts II and III consider particular relations, and some specific results are obtained determining the degree in the hierarchy of a number of such relations. They are for the most part of the following kind: the relation P holds for a nonnegative integer n when n is the Gödel number of a general recursive function belonging to a family P of functions. These families of functions are in Part II the functions giving non-terminating binary expansions for recursive real numbers in certain classes of real numbers (e.g., rational, algebraic). In Part III they are the functions whose ranges are in certain classes of r.e. sets.

H. G. Rice (Pittsburgh, Pa.).

Uspenskiĭ, V. A. On the theorem of uniform continuity. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 99–142. (Russian)

This paper involves a constructive approach to the theory of functions defined in Baire space (J). The points of J are the infinite sequences of natural numbers. A point $(n_0, n_1, \cdots)$ is identified with the function f such that, for all i, $f(i)=n_i$. The "constructive analog" of J is the subset J^* consisting of the general recursive functions. The distance between distinct points f and g of J is $1/(k+1)$, where k is the least k such that $f(k)\neq g(k)$. The Baire intervals are the spherical neighborhoods defined with this metric. Open subsets are defined in the usual way in terms of Baire intervals. The Baire intervals map 1-1 onto the set N_2 of finite or empty sequences of natural numbers. A subset of J is "constructively open" if it is representable in the form $\delta_{\zeta(0)}\cup\delta_{\zeta(1)}\cup\cdots$, where ζ is a completely defined, computable function with values in N_2 and $\delta_{\zeta(t)}$ is the Baire interval corresponding to $\zeta(t)$. Closed ("constructively closed") subsets of J are complements of open (constructively open) subsets.

The constructive notion of compactness is obtained, not from the direct definition of compactness, but instead from a defining predicate which is equivalent for Baire space. Thus, a subset of J is compact if and only if it is bounded by a point of J (i.e. if there exists a point g such that, for every point f of the subset, $f(i)\leq g(i)$ $(i=0, 1, \cdots)$); a subset is "constructively compact" if it is bounded by a point of J^*. Here and elsewhere, as the author instructively points out, various equivalent defining predicates for a given notion may be chosen as starting-point for constructivization, but the resulting constructive analogs of the notion may be non-equivalent. Some of the author's choices coincide with those of Kuznecov and Trahtenbrot [Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 897–900; MR **17**, 1039].

A subset of J is compact in itself if and only if it is compact and closed. Correspondingly, a subset is "constructively compact in itself" if it is constructively compact and constructively closed. However, the constructive analog of a "compact" (i.e. a subset compact in itself) is the intersection with J^* of a subset constructively compact in itself; such a "constructive compact" is not necessarily compact or closed in the usual sense.

A constructive notion of continuity for functions defined in J is given for N-valued functions (with values in the set N of natural numbers) and for J-valued functions. Continuity is first characterized in terms of "regularity" — a notion more suitable for constructivization. A certain class of N-valued functions in N_2 is defined in terms of a partial ordering of N_2. These functions are mapped onto the "regular" N-valued functions in J. An N-valued function in J is continuous if and only if it can be extended to a regular N-valued function. Correspondingly, an N-valued function is "constructively regular" if it corresponds in the above mapping to a computable function; "constructively continuous" if it can be extended to a constructively regular function. Similarly, regularity and constructive regularity, continuity and constructive continuity are defined for J-valued functions. A "constructive continuous N-valued (J-valued) function" is a constructively continuous N-valued (J-valued) function defined only in J^*.

In an interesting digression, the author submits an interpretation, based on constructively regular functions, of some intuitionistic function-theoretic notions of H. Weyl [Math. Z. 10 (1921), 39–79].

A function is uniformly continuous if and only if it has a "regulator". (A regulator, for a function f from space X with metric ρ_X to space Y with metric ρ_Y, is any function γ with arguments and values in N such that, for every positive n, $\rho_X(x, y) < 1/\gamma(n)$ implies $\rho_Y(f(x), f(y)) < 1/n$.) A function is "constructively uniformly continuous" if it has a general recursive regulator.

Every continuous N-valued (J-valued) function defined on a compact is uniformly continuous. On the other hand, the constructively analogous assertions — "Every constructive continuous N-valued (J-valued) function defined on a constructive compact is constructively uniformly continuous" — are both false. An N-valued counterexample, due to Kleene [Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 1, Amer. Math. Soc., Providence, R.I., 1952, pp. 679–685; MR 13, 422], is defined on the set of general recursive functions with all values either 0 or 1, and is not uniformly continuous even in the usual sense. From this example, a J-valued counterexample is produced. The author presents a counterexample of his own, simpler than Kleene's.

The foregoing notions are extended by defining a "generalized Baire space $\bar{J}$", consisting of the infinite sequences whose members belong to $N \cup \{O\}$, where O is an object not in N. The points of $\bar{J}$ are identified in the obvious way with the partial N-valued functions in N. The $\bar{J}$-valued functions in $\bar{J}$ become operators (i.e. functionals). A system of neighborhoods is defined so that $\bar{J}$ becomes a topological space in which J is imbedded. Definitions of regularity, continuity and their constructive analogs are similar to the preceding ones. The author identifies the computable operators with the constructively continuous $\bar{J}$-valued functions on $\bar{J}$; an alternative approach is due to Kleene [Introduction to metamathematics, Van Nostrand, New York, 1952; MR 14, 525].

G. F. Rose (Santa Monica, Calif.).

Rice, H. G. On the relative density of sets of integers. Proc. Amer. Math. Soc. 8 (1957), 320–321.

This paper is based on Yu. T. Medvedev's paper [Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 211–214; MR 18, 272] and the author's paper [Trans. Amer. Math. Soc. 83 (1956), 277–300; MR 18, 712]. It deals with the collection ε of all numbers (i.e., non-negative integers) and with sets (i.e., collections of numbers). The strictly increasing function which ranges over an infinite set is called the principal function of that set. A function $a(x)$ from ε into ε is called recursively bounded if $a(x) \leq f(x)$ for some recursive function $f(x)$. The class of all infinite sets whose principal function is recursively bounded is denoted by B. Let for any set σ and any number n, the cardinality of the set $\{x | x \in \sigma \text{ and } x \leq n\}$ be denoted by $\sigma[n]$. A set α_2 is at least as dense as α_1 (written: $\alpha_1 \leq \alpha_2$), if $\alpha_1[n] \leq \alpha_2[g(n)]$ for some recursive function $g(n)$. The sets α_1 and α_2 have the same degree of density if $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$. These notions were introduced by Medvedev. Clearly, $\sigma \leq \varepsilon$ for every set σ.

The author first proves the following lemma: if the infinite sets α_1 and α_2 have the principal functions $h_1(x)$ and $h_2(x)$ respectively, then $\alpha_1 \leq \alpha_2$ if and only if $h_2(x) \leq g(h_1(x))$ for some recursive function $g(x)$. Thus B consists exactly of those sets which have the same degree of density as ε; the class B constitutes therefore the highest degree of density. Using this lemma the author proves the following theorem: the permutation $p(x)$ of ε is recursively bounded if and only if $\sigma \leq p(\sigma)$ for every set σ. This implies that B is closed under recursively bounded permutations of ε.

J. C. E. Dekker (Princeton, N.J.).

Barker, C. C. H. Some calculations in logic. Math. Gaz. 41 (1957), 108–111.

This note illustrates a method for dealing with problems in elementary two-valued logic based on the definition given by M. H. Stone [Trans. Amer. Math. Soc. 40 (1936), 37] of a Boolean ring .The method is then extended to deal with three-valued logic.

Kalicki, J. On equivalent truth-tables of many-valued logics. Proc. Edinburgh Math. Soc. (2) 10, 56–61 (1954).

Kanger, Stig. A note on partial postulate sets for propositional logic. Theoria 21 (1955), 99–104.

A set F of axioms for the propositional calculus is given here satisfying the following 6 conditions: (i) each axiom of F is independent of the remaining axioms, (ii) F yields the theorems of the classical sentential calculus with $\supset$, &, $\vee$ and $\neg$ as primitive connectives, (iii) there is a subsystem H of f with $n-1$ axioms that yields the theorems of Heyting's intuitionistic sentential calculus, (iv) there is a subsystem J of H with $n-2$ axioms that yields the theorems of Johansson's minimal calculus, (v) each axiom of F contains $\supset$ and no axiom contains more than two different connectives, (vi) for each selection γ of connectives (from $\supset$, &, $\vee$ and $\neg$) that includes implication, and each choice S from F, H and J, there is a subsystem S_γ of S, the axioms of which contain no other connectives than those of γ and which yields all theorems derivable from S that contain connectives of γ only. The proof makes strong use of some results in "A theory of formal deducibility" [Univ. of Notre Dame, 1950; MR 11, 487] by H. B. Curry.

L. N. Gál (Ithaca, N.Y.).

Schütte, Kurt. Ein System des verknüpfenden Schliessens. Arch. Math. Logik Grundlagenforsch. 2 (1956), 55–67.

Much of the importance of Gentzen's formulation of the first order predicate calculus is due to the fact that the logical rules used are apparent in the deduction. In many applications the structural rules of inference and those rules in which the conclusion is weaker that the hypothesis (e.g. &→, ∨→) are of lesser importance, so that a system in which these play a subsidiary role is of interest. Such a system is presented here and its completeness is proved. This proof is extended to a proof of the completeness of number theory based on this system. This formulation allows both proofs to be presented rather neatly.

L. N. Gál (Ithaca, N.Y.).

Thiele, Helmut. Eine Axiomatisierung der zweiwertigen Prädikatenkalküle der ersten Stufe, welche die Implikation enthalten. Z. Math. Logik Grundlagen Math. 2 (1956), 93–106.

It is the main purpose of this paper to establish semantic completeness for certain wide classes of systems of axioms within the two-valued lower predicate calculus. The inclusion of propositional connectives other than the usual ones is permitted. The axioms specified for these connectives arise in a natural manner from the truth tables defining them. It is shown that the available methods for establishing completeness are sufficient to cope with the systems considered here.

A. Robinson.

Takeuti, Gaisi. On Skolem's theorem. J. Math. Soc. Japan 9 (1957), 71–76.

Dem Löwenheim-Skolemschen Satz, dass jedes erfüllbare Axiomensystem (aus der elementaren Prädikatenlogik mit Identität) ein höchstens abzählbares Modell besitzt, entspricht der reinsyntaktische Satz, dass ein konsistentes Axiomensystemo konsistent belibt, wenn mit einem neuen Funktionssymbol f_0 das Axiom

$$\forall x \exists y (y \in \omega \wedge f_0(y) = x)$$

hinzugefügt wird. ω ist hierbei die Menge der natürlichen Zahlen: zu Γ_0 muss also zunächst ein genügend reichhaltiges Axiomensystem der Arithmetik hinzugenommen werden. Verfasser beweist diesen Satz metamathematisch auf der Grundlage seiner früheren Arbeiten über "a generalized logic calculus", Jap. J. Math. 23 (1953), 39–96; 24 (1954), 149–156; MR 17, 701]. *P. Lorenzen.*

★ **Шанин, Н.А. [Šanin, N. A.] О некоторых логических проблемах арифметики. [On some logical problems of arithmetic.]** Trudy Mat. Inst. im. Steklov. no. 43. Izdat. Akad. Nauk SSSR, Moscow, 1955. 112 pp. 5.20 rubles.

This work is concerned with mapping from a classical "logico-arithmetic system" into a constructive (intuitionist) one. [For an abstract see Dokl. Akad. Nauk SSSR (N.S.) 93 (1953), 779–782; MR 15, 593; also Mostowski, J. Symb. Logic 19 (1954), 297–298.] Here a logico-arithmetic system is an applied predicate calculus with individual variables ranging over natural numbers; one basic predicate, equality, is allowed, and no propositional or predicate variables; there is a constant 0, and the successor, sum, and product functions; and the axioms, besides those derived from the schemes of the logical calculus, express the Peano postulates, the properties of equality, and the recursive definitions of the sum and product. Reference is made to the work of Kleene and Nelson on recursive realizability [see § 82 of Kleene's book, Introduction to metamathematics, Van Nostrand, New York, 1952; MR 14, 525]. The author's constructive system Σ is very similar to that of Kleene and Nelson; the classical system Σ^+ is obtained by adding all instances of the law of excluded middle. About half of the book is devoted to an exposition of fundamentals which is of some interest from the standpoint of philosophy of formal methods. The last two chapters are devoted to the study of embedding (pogružayuščie; literally "immersing") operators α, where α maps any formula P into an image αP such that, if P is valid in Σ^+, αP is valid in Σ, and from a proof of αP in Σ one can find a proof of P in Σ^+ (usually because P and αP are equivalent in Σ^+). This subject is the same as §81 of Kleene (loc. cit.); but several operations not treated there are considered here (and vice-versa in a few cases). A chapter is devoted to regular embedding operators, i.e. those such that $P \supset \alpha P$ is recursively realizable (and, in all cases considered, derivable in Σ). The operators of Kolmogoroff, Gödel, etc. did not have this property. Several examples of such transformations are considered; some simple ones make considerable disturbances in P, while others which are more complex, make less disturbances. The interrelations between these are investigated, and full proofs are given. *H. B. Curry.*

Henkin, Léon. The algebraic structure of mathematical theories. Bull. Soc. Math. Belg. 7 (1955), 131–136.

This is an informal account of some topics which have been expounded recently in considerable detail in a book by the same author [La structure algébrique des théories mathématiques, Gauthier-Villars, Paris, 1956; MR 18, 272]. Mention is made of the theory of expressions, of Boolean algebras, corresponding to the theory of propositions, and finally of the need for cylindrical algebras to correspond to the lower predicate calculus.

A. Robinson (Toronto, Ont.).

Ehrenfeucht, A. Application of games to some problems of mathematical logic. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 35–37, IV, (Russian summary)

This is a preliminary report and the proofs of the theorems contained in it will be published in Fund. Math. It contains a necessary and sufficient condition for elementary indiscernibility of models, a sufficient condition for the indiscernibility of models by means of finite sets and three theorems, the second of which is the following: If player II has a winning method in $H_n(M', M'')$ for every n, then M' and M'' are indiscernible by finite sets. Here $H_n(M', M'')$ is a game in which the two players I and II make n moves, and M' and M'' are two models for an elementary theory T which has no terms and a finite number of predicates only. The set of these predicates is denoted by P. In every move the player I points out an arbitrary finite number m of elements either from $|M'|$ or from $|M''|$, where $|M|$ denotes the set of individuals of M. Player II has to match every individual pointed out by I with an individual in the other model. Player II wins if for any predicate $\alpha \in P$ and any finite sequence $k_1, \cdots, k_{a(\alpha)}$, where $1 \leq k_i \leq n$, $\mathrm{stsf}_{M'}\alpha(a'_{k_1}, \cdots, a_{ka(\alpha)})$ if, and only if, $\mathrm{stsf}_{M''}\ \alpha(a_{k_1}'', \cdots, a_{ka(\alpha)}'')$. All notations concerning theories and models are those used by Ehrenfeucht and Mostowski [Fund. Math. 43 (1956), 50–68; MR 18, 863]. *B. Germansky* (Jerusalem).

Sierpinski, W. Sur une propriété de la droite équivalente à l'hypothèse du continu. Ganita 5 (1954), 113–116 (1955).

Proposition H: $2^{\aleph_0} = \aleph_1$. Proposition P: There exists a correspondence that associates with each point p of the line a countable set of segments one of whose endpoints is p, and such that every segment is associated with some point. Proposition P_1: The plane is the union of two sets A and B such that A meets every horizontal line in a countable set and B meets every vertical line in a countable set. The author proves that H is equivalent to P, the proof that P implies H requiring the axiom of choice. He then proves, without the intervention of either H or the axiom of choice, that P is equivalent with P_1. [A direct proof of the equivalence of H with P_1 was given by the author in Bull. Internat. Acad. Polon. Sci. Lett. Cl. Sci. Math. Nat. Sér. A. 1919, 1–3.] *L. Gillman.*

Noguera Barreneche, Rodrigo. Résolution générale du problème de trichotomie des nombres cardinaux. Stvdia. Rev. Univ. Atlantico 1 (1956), no. 10, 73–76.

★ **Соминский, И.С. [Sominskiĭ, I. S.] Метод математической индукции. [The method of mathematical induction.]** 4th ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 48 pp. 0.75 rubles.

A German translation was reviewed in MR 16, 104.

Rudin, Mary Ellen. A subset of the countable ordinals. Amer. Math. Monthly 64 (1957), 351.

There exists a subset of the ordered set of all countable ordinals that intersects every uncountable closed set but contains no uncountable closed set. *L. Gillman.*

See also: Vasilache, p. 5; Bini, p. 53.

ALGEBRA

Benedicty, Mario. Alcune applicazioni della nozione di insieme quoziente. Archimede 9 (1957), 1–5.

Expository remarks about equivalence classes.

Cartan, Henri. Strutture algebriche. Archimede 9 (1957), 10–19.

An expository lecture.

★ **Vasilache, Sergiu. Elemente de teoria multimilor şi a structurilor algebrice. [Elements of the theory of sets and of algebraic structures.]** Editura Academiei Republicii Populare Romîne, 1956. 236 pp. 11.65 Lei.

There are 3 chapters: algebra of propositions, theory of sets, algebraic structures. Transfinite numbers, cardinal and ordinal, are dealt with in the second chapter. Among the topics of the third chapter are algebraic operations (interior and exterior), groups, groups with operators, rings, ideals, fields, general theorems about homomorphism, vector spaces, duality, linear equations, hypercomplex systems.

Combinatorial Analysis

Tietze, Heinrich. Ein Prinzip für Schachturnier-Tabellen. Archimedes 8 (1956), 41–43.

Geometric formulation of a simple rule, attributed to the chess master Przepiorka (1905) for constructing a round-robin tournament schedule for $n>2$ players involving a minimal number of rounds and allowing each player to play white in as near half of his games as possible [see Kraitchik, La mathématique des jeux $\cdots$, Stevens, Bruxelles, 1930, p. 544]. *L. Moser* (Edmonton, Alta).

Yacoub, K. R. On semi-special permutations. I. Proc. Glasgow Math. Assoc. 3 (1956), 18–35.

Suppose that a semi-special permutation π [for definitions, see Yacoub, same Proc. 2 (1955), 116–123; MR 17, 11] on $[n]$, the set $\{1, 2, \cdots, n\}$, is not linear; that is, π does not have the form $\pi x\equiv rx \pmod n$, r prime to n, for every $x\in[n]$. The author shows that there is a proper divisor s of n such that $\pi_s x\equiv\pi(s+x)-\pi(x)=\pi(x)$ for all $x\in[n]$ and that π induces a linear permutation on $[s]$. The maximum s with these properties is called the principal number of π. He shows, for instance, that a non-linear semi-special π on $[n]$, with principal number s, where π induces the identity on $[s]$, can be written in the form $\pi x\equiv x+s\lambda((1-\omega^x)/(1-\omega)) \pmod n$, where $(\lambda, n/s)=1$, and $\omega_s\equiv1$, $\omega\not\equiv1 \pmod{n/s}$. If the induced map on $[s]$ is not the identity, the situation is much more involved, but the author manages to find necessary and sufficient conditions for the existence of non-linear semi-special permutations. He determines the structure of such permutations on $[p^2]$. On $[pq]$ where p does not divide $q-1$, the semi-special permutations are all linear. In the case $p|(q-1)$, the author is able to describe all non-linear semi-special permutations. *F. Haimo.*

See also: Bennett, p. 16; Behrens, p. 55; Chakravarti, p. 74; Sprott, p. 76.

Elementary Algebra

Bellman, Richard. On the arithmetic-geometric mean inequality. Math. Student 24 (1956) 233–234 (1957).

The object of this note is to present yet another proof of this inequality based upon the functional equation technique of the theory of dynamic programming. *From the introduction.*

Majó Torrent, J. Note on the extension to the complex field of the arithmetic and the geometric mean. Gac. Mat., Madrid (1) 8 (1956), 195–198. (Spanish)

Consider the $(n+1)$ complex numbers $z_k=r_ke^{i\theta_k}$ $(k=1, 2, \cdots, n)$, $z=re^{i\theta}$, where $n\theta=\sum\theta_k$. If r is the arithmetic (geometric) mean of the r_k, the author calls z the "polar" arithmetic (geometric) mean of the z_k and discusses certain properties of these means.

Linear Algebra

★ **Murdoch, D. C. Linear algebra for undergraduates.** John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1957. xi+239 pp. $5.50.

There are 9 chapters: vectors and vector spaces; matrices, rank and systems of linear equations; further algebra of matrices; further geometry of real vector spaces; transformation of coordinates in a vector space; linear transformations in a vector space; similar matrices and diagonalization theorems; reduction of quadratic forms; vector spaces over the complex field. There are 2 appendixes on abstract vector fields and solid analytic geometry. Many problems are provided, with answers to most of them.

Suschowk, Dietrich. Über die gegenseitige Lage zweier linearer Vektorräume. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. 1956, 15–22 (1957).

Let $\mathscr{T}_i$ $(i=1, 2)$ be an m_i-dimensional subspace of n-dimensional unitary space $\mathscr{U}_n$, and let P_i be the projection on $\mathscr{T}_i$. The Hermitian transformation P_1P_2 has eigenvalues $1\geqq\lambda_1\geqq\cdots\geqq\lambda_n\geqq0$, where

$$(1)\qquad \lambda_{m+1}=\cdots=\lambda_n=0$$

if $m=\min(m_1, m_2)<n$, and

$$(2)\qquad \lambda_1=\cdots=\lambda_{m_1+m_2-n}=1$$

if $m_1+m_2>n$. The number of zeros among $\lambda_1, \cdots, \lambda_{m_1}$ is the dimension of $\mathscr{T}_1\cap\mathscr{T}_2^{\perp}$, and the number of λ's which are 1 is the dimension of $\mathscr{T}_1\cap\mathscr{T}_2$.

The author associates with each pair of subspaces $\mathscr{T}_1$, $\mathscr{T}_2$ the ordered real vector $(\lambda_1, \cdots, \lambda_n)$, which is denoted by $J(\mathscr{T}_1, \mathscr{T}_2)$. ($J(\mathscr{T}_1, \mathscr{T}_2)=J(\mathscr{T}_2, \mathscr{T}_1)$, since P_1P_2 and P_2P_1 have the same eigenvalues.) He shows that if $\mathscr{S}_i$ $(i=1, 2)$ are subspaces such that $\dim\mathscr{S}_i=\dim\mathscr{T}_i$ then $J(\mathscr{T}_1, \mathscr{T}_2)=J(\mathscr{S}_1, \mathscr{S}_2)$ if and only if there exists a unitary transformation Φ of $\mathscr{U}_n$ such that $\Phi\mathscr{T}_i=\mathscr{S}_i$. The author also shows that, given real numbers $1\geqq\lambda_1\geqq\cdots\geqq\lambda_n\geqq0$, and positive integers $m_1, m_2\leqq n$, where the λ's satisfy (1) and (2), there exist m_i-dimensional subspaces $\mathscr{T}_i$ $(i=1, 2)$ of $\mathscr{U}_n$ such that $J(\mathscr{T}_1, \mathscr{T}_2)=(\lambda_1, \cdots, \lambda_n)$. *B. N. Moyls.*

Dieudonné, Jean. Pseudo-discriminant and Dickson invariant. Pacific J. Math. 5 (1955), 907–910.

To each nondegenerate quadratic form Q over a field K of characteristic 2 is attached a class $\Delta(Q)$ mod $\wp K$ [see Arf, J. Reine Angew. Math. **183** (1941), 148–167; MR **4**, 237]. The author gives a very short and simple proof of the following statement and he connects it with a result of L. E. Dickson. If $Q_1(x)=Q(u(x))$, where u is a symplectic transformation, and if $\Delta(Q)$ and $\Delta(Q_1)$ are computed with respect to the same symplectic basis, then

$$\Delta(Q_1)=\Delta(Q)+\wp D(u),$$

where $D(u)$ is the so called Dickson invariant of u.

C. Arf (Istanbul).

Allen, H. S. Commutative rings of linear transformations and infinite matrices. Quart. J. Math. Oxford Ser. (2) 8 (1957), 39–53.

Let R and R' be left and right vector spaces respectively over a "sfield" Δ. Assume that R and R' are dual vector spaces relative to a non-degenerate bilinear form (x, y) and denote by $L(R'|R)$ the ring of all those linear transformations T in R which possess transposes T' in R'. If M is a subspace of R (of R') denote by $j(M)$ the subspace of R' (of R) consisting of all elements $y \in R'$ $(x \in R)$ such that $(x, y)=0$ for every $x \in M$ $(y \in M)$. Let Φ denote the center of Δ. Let M, N be subspaces of R, R' respectively and denote by $\Phi(N|M)$ the ring of all $T \in L(R'|R)$ for which there exists $a \in \Phi$ such that $xT=ax$ for all $x \in M$, and $xT'=ya$ for all $y \in N$. A sample of the results obtained is the following: (1) The centralizer $C[\Phi(N|M)]$ of $\Phi(N|M)$ in $L(R'|R)$ is equal to $\Phi(j(M)|j(N))$. (2) $\Phi(N|M)$ is commutative if $M \supseteq j(N)$. (3) If $\Phi(M|N)$ is commutative, then either $M \supseteq j(N)$ or Δ is commutative and M, N are hyperplanes. (4) $\Phi(M|N)$ is a maximal commutative subring of $L(R'|R)$ if and only if $N=j(M)$. (5) Set $\Phi(M)=\Phi(j(M)|M)$, $\theta(M)=$ the Jacobson radical of $\Phi(M)$. Then $(\theta(M))^2=0$ and $\Phi(M)/\theta(M)$ is isomorphic to Φ. (6) Let $L_M(R'|R)$ be the ring of all elements of $L(R'|R)$ which leave a closed subspace M $(M\neq 0)$ of R invariant. If B is a proper 2-sided ideal in $L_M(R'|R)$, then $C(B)$ is commutative and is maximal commutative in $L(R'|R)$ if and only if $B \subseteq \theta(M)$.

The author applies his results for the general case to obtain a number of results for certain rings of infinite matrices. *C. E. Rickart* (New Haven, Conn.).

Taussky, Olga. Commutativity in finite matrices. Amer. Math. Monthly 64 (1957), 229–235.

This paper, an invited address before the Mathematical Association, gives a survey on newer results which depend on the commutativity of matrices or which hold if the commutativity is replaced by a weaker condition. In particular, commutators of higher order, matrices with property P, matrices with property L, matrices in $A+zB$ with multiple eigenvalues, and diagonable pencils of matrices are considered. An extensive bibliography is added. *A. Brauer* (Chapel Hill, N.C.).

Hodges, John H. Weighted partitions for skew matrices over a finite field. Arch. Math. 8 (1957), 16–22.

For $q=p^n$ $(p>2)$, put $e(\alpha)=e^{2\pi i t(\alpha)/p}$,

$$t(\alpha)=\alpha+\alpha^p+\cdots+\alpha^{p^{n-1}} \quad (\alpha \in \mathrm{GF}(q)).$$

Also if $A=(\alpha_{ij})$ is a square matrix, define $\sigma(A)=\sum \alpha^i{}_i$. Define

$$(*) \qquad S=S(B, U, A)=\sum_X e\{\sigma(U'X+XU')\},$$

where A is skew and non-singular of order $2m$, B is skew of order t, $U=U(2m, t)$ is arbitrary and the summation is over all $X=X(2m, t)$ such that $X'AX=B$. The problem is to evaluate S. The writer has discussed the analogous problem for symmetric and for general square matrices in previous papers [Math. Z. **66** (1956), 13–24; Duke Math. J. **24** (1956), 545–552; MR **18**, 643, 113]. To evaluate S, the sum in (*) is broken into three parts S_1, S_2, S_3. The first sum S_1 is easily evaluated explicitly, while S_2 and S_3 are expressed in terms of generalized Kloosterman sums

$$K_{2r}(A, B)=\sum_C e\{-\sigma(BC+C^{-1}A)\},$$

the summation extending over non-singular skew C of order $2r$. Thus the sum S is evaluated in all cases; in particular if U is non-singular of order $2m$, then

$$S=q^{m(m+1)}K_{2m}(B, A^*),$$

where $A^*=U'A^{-1}U$.

In the concluding section of the paper, a number of properties of $K_{2r}(A, B)$ are obtained. *L. Carlitz.*

Fan, Ky; and Pall, Gordon. Imbedding conditions for Hermitian and normal matrices. Canad. J. Math. 9 (1957), 298–304.

Let A, B be two square matrices with complex coefficients of orders n and m respectively, $n \geq m$. The matrix B is said to be imbeddable in A if there exists a unitary matrix U such that U^*AU contains B as a principal submatrix. For Hermitian matrices A and B with eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \cdots \geq \beta_m$, the authors show that B is imbeddable in A if and only if $\alpha_i \geq \beta_i$ and $\alpha_{n-i+1} \leq \beta_{m-i+1}$ for $1 \leq i \leq m$. The necessity part is known [cf. Hamburger and Grimshaw, Linear transformations in n-dimensional vector space, Cambridge, 1951; MR **12**, 836]. For the sufficiency part the essential case is $m=n-1$. This case is readily extended to the case $m<n-1$ by using a chain of Hermitian matrices, with orders increasing by unity, such that each is imbeddable in the next. For symmetric matrices this result is valid when U is required to be orthogonal.

The authors generalize the case $m=n-1$ to normal matrices. They show by an example that the chain process used for Hermitian matrices to obtain the case $m<n-1$ will not, in general, work for normal matrices. Their result is as follows: Let A, B be normal matrices of orders n and $n-1$, respectively; let the eigenvalues of A and B be $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_{n-1}$, numbered so that $\alpha_1, \cdots, \alpha_q$ are each distinct from $\beta_1, \cdots, \beta_{q-1}$, while $\alpha_j=\beta_{j-1}$ for $q+1 \leq j \leq n$. Then B is imbeddable in A if and only if the points $\alpha_1, \cdots, \alpha_q, \beta_1, \cdots, \beta_{q-1}$ are distinct, collinear, and such that every segment of this line limited by two adjacent α_i's contains one β_j. *B. N. Moyls.*

Bellman, Richard. Notes on matrix theory. IX. Amer. Math. Monthly 64 (1957), 189–191.

The following theorem is proved.

Let A and B be two positive definite matrices of order n, and let $C=\lambda A+(1-\lambda)B$, for $0 \leq \lambda \leq 1$. For each $j=1, 2, \cdots, n$, let $A^{(j)}$ denote the principal submatrix of A obtained by deleting the first $(j-1)$ rows and columns, (in particular, $A^{(1)}=A$). Let $B^{(j)}$, $C^{(j)}$, have similar meanings. If $k_1, k_2, \cdots, k_n$ are n real numbers such that $\sum_{i=1}^{j} k_i \geq 0$, $j=1, 2, \cdots, n$, then

$$\prod_{j=1}^{n} |C^{(j)}|^{k_j} \geq \prod_{j=1}^{n} |A^{(j)}|^{\lambda k_j} |B^{(j)}|^{(1-\lambda)k_j}.$$

Brauer, Alfred. The theorems of Ledermann and Ostrowski on positive matrices. Duke Math. J. **24** (1957), 265–274.

It is known that the absolute greatest characteristic root ω of a positive matrix does not exceed the maximum row sum and, in fact, is smaller than it unless all row sums coincide. Ledermann [J. London Math. Soc. **25** (1950), 265–268; MR **12**, 312] and later Ostrowski [ibid. **27** (1952), 253–256; MR **14**, 126] gave upper and lower bounds for ω in terms of the maximum and minimum row sums, their quotient and the minimum element of the matrix. Now even better bounds are given, as functions of the same quantities, and are shown to be even best possible. Also two bounds are obtained which are often even better by using the minimum elements of certain columns.

It is further known that ω exceeds the diagonal elements of the matrix. Let a be the largest diagonal element. A lower bound is given for $\omega-a$ in terms of the maximum and minimum row sums, the minimum off-diagonal element and the sum of the elements of the kth row, where $a=a_{kk}$. *O. Taussky-Todd* (Pasadena, Calif.).

Medlin, Gene W. On limits of the real characteristic roots of matrices with real elements. Proc. Amer. Math. Soc. **7** (1956), 912–917.

The following theorem is proved. Let $A=(a_{\kappa\lambda})$ be a square matrix of order n with real elements and ω a real characteristic root. Set

$$M_{\kappa\lambda}^*=\sum_{\nu=1}^{n}|a_{\kappa\lambda}a_{\lambda\nu}+a_{\lambda\kappa}a_{\kappa\nu}|,$$

$$m_{\kappa\lambda}^*=\sum_{\nu=1}^{n}|a_{\kappa\lambda}a_{\lambda\nu}-a_{\lambda\kappa}a_{\kappa\nu}|,$$

$$M_{\kappa\lambda}=\max(M_{\kappa\lambda}^*, m_{\kappa\lambda}^*).$$

In $\sum_\nu a_{\kappa\nu}a_{\lambda\nu}$ $(1\leqq\nu\leqq n;\ \nu\neq\kappa,\lambda)$, let $W_{\kappa\lambda}^*$ denote the sum of the positive terms, $w_{\kappa\lambda}^*$ the sum of the negative terms, and

$$W_{\kappa\lambda}=\max(W_{\kappa\lambda}^*, w_{\kappa\lambda}^*).$$

Set

$$Q_{\kappa\lambda}=M_{\kappa\lambda}+W_{\kappa\lambda}+\sum_{\nu=1}^{n}|a_{\kappa\nu}a_{\lambda\mu}+a_{\kappa\mu}a_{\lambda\nu}|,$$

where $\nu<\mu$, and $\nu\neq\kappa,\lambda$, and $\mu\neq\kappa,\lambda$. Then each real characteristic root ω must satisfy at least one of the $n(n-1)/2$ inequalities

$$|(\omega-a_{\kappa\kappa})(\omega-a_{\lambda\lambda})-a_{\kappa\lambda}a_{\lambda\kappa}|\leqq Q_{\kappa\lambda}\quad(\kappa,\lambda=1,2,\cdots,n;\ \kappa\neq\lambda).$$

This result is not as sharp as a theorem of the reviewer [J. Reine Angew. Math. **192** (1953), 113–116; MR **15**, 496], but is often simpler for the applications. *A. Brauer.*

Lehmer, D. H. On certain character matrices. Pacific J. Math. **6** (1956), 491–499.

Let p be an odd prime and let $\chi(n)$ be the Legendre symbol which is usually denoted by $\left(\frac{n}{p}\right)$ The author discusses two types of square matrices $A=(a_{ij})$ of dimension p which arise from the theory of exponential sums, namely

(i) $a_{ij}=a+b\chi(i)+c\chi(j)+d\chi(ij)$,

(ii) $a_{ij}=e+\chi(\alpha+i+j)$,

where a, b, c, d and e are any constants and α is an integer. In both cases explicit formulae are given for the general power A^k, including A^{-1} if it exists, and for the latent roots of A. The paper also contains an elegant proof of a quadratic relation for the values $\chi(n)$ which was first published by E. Jacobstahl [Dissertation, Berlin, 1906]. *W. Ledermann* (Manchester).

Caianiello, E. R. Proprietà di Pfaffiani e Hafniani. Ricerca, Napoli **7** (1956), 25–31.

Dans cet article l'Auteur rassemble — sans démonstration — certaines propriétés des Pfaffiens et des Haffniens. Il donne en particulier des règles de développement des Pfaffiens. Les Haffniens — introduits par l'A. — sont définis à partir des Pfaffiens de la même manière que les permanents sont définis à partir des déterminants: on remplace tous les signes moins qui apparaissent dans le développement par des signes plus.

Les algèbres de Grassmann et de Clifford jouent un rôle primordial dans cet exposé. L'auteur indique, en particulier, comment la considération de certaines dérivations permettent de se ramener de l'algèbre de Clifford à celle de Grassmann.

L'Auteur utilise de très nombreux abus de langage (à la manière des physiciens). Des fautes d'impression rendent la lecture de cet article malaisée. *G. Papy.*

Carlitz, L. A determinant. Amer. Math. Monthly **64** (1957), 186–188.

A generalization of the theorem of Nesbitt [Niblett, same Monthly **59** (1952), 171–174; MR **13**, 813].

See also: Lichnerowicz, p. 19; Steinberg, p. 55; Healy, p. 64; Zięba, p. 106.

Polynomials

Selmer, Ernst S. On the irreducibility of certain trinomials. Math. Scand. **4** (1956), 287–302.

It is shown that the polynomial x^n-x-1 is irreducible for all n. The polynomial x^n+x+1 is irreducible for $n\not\equiv2\pmod 3$. When $n\equiv2\pmod 3$ it has the factor x^2+x+1. Certain other trinomial polynomials are also considered. *O. Ore* (New Haven, Conn.).

Gasapina, U. Le equazioni di quarto grado. Period. Mat. (4) **35** (1957), 44–55.

An expository account of parts of the prehistory of the Galois theory.

See also: Jankowski, p. 22.

Partial Order, Lattices

★ **Morgado, José. Elementos de algebra moderna: reticulados, sistemas parcialmente ordenados. [Elements of modern algebra: lattices, partially ordered systems.] Vol. I. (Fundamental concepts).** Porto, 1956. 120 pp.

The first volume contains general information and closes with an introduction to the theory of representation. The author states that the second volume will deal with equivalence relations and ideals, with applications to the theory of representation. The style is expository and the book can be read without especial mathematical preparation.

Pinsker, A. G. A lattice characterization of function spaces. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 226–229. (Russian)

This paper announces without proof a complicated set

of necessary and sufficient conditions for a lattice to be isomorphic to the lattice of all real-valued continuous functions on a compact Hausdorff space. This solves in a manner of speaking a problem raised by G. Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed.; New York, 1948; MR **10**, 673; see also the authors earlier announcement, Dokl. Akad. Nauk SSSR (N.S.) **99** (1954), 503–505; MR **16**, 560]. *E. Hewitt.*

Lamperti, John. A note on autometrized Boolean algebras. Amer. Math. Monthly **64** (1957), 188–189.

Concise proofs of two theorems by D. Ellis [Canad. J. Math. 3 (1951), 145–147; MR **13**, 377].

See also: Jaffard, p. 13; Berberian, p. 47.

Fields, Rings

Whaples, G. Algebraic extensions of arbitrary fields. Duke Math. J. **24** (1957), 201–204.

Two questions are raised and answered: 1. Is the condition that a field has no extension of degree divisible by p equivalent to the condition that it has no extension of degree p? The answer is 'no'. It is even shown that to any integer n there is a field which has algebraic extensions of degree divisible by n, but none of degree $\leq n$. 2. If a field has a cyclic extension of degree p does it have a cyclic extension of degree p^γ for every γ?. The answer is 'yes', unless $p=2$ and the field is ordered and every sum of squares is a square. A cyclic extension of degree 4 ensures one of degree 2^γ for all γ. The proof splits up into the case when p divides the characteristic and when it does not and according to the roots of unity contained in the ground field. O. Taussky-Todd.

Ono, Takashi. On algebraic groups defined by norm forms of separable extensions. Nagoya Math. J. **11** (1957), 125–130.

Let L be a finite separable algebraic extension of the infinite field K. The author studies the group of linear transformations of the vector space L over K for which the norm function of L/K is semi-invariant (resp. invariant). This group, which is algebraic in the sense of Chevalley, is generated by the K-automorphisms of the field L and the scalar multiplications by nonzero elements of L (resp. elements of L of norm 1). *M. Rosenlicht.*

Beyer, Gudrun. Über relativ-zyklische Erweiterungen galoisscher Körper. J. Reine Angew. Math. **196** (1956), 34–58.

Beyer, Gudrun. Über eine Vermutung von Hasse zum Erweiterungsproblem galoisscher Zahlkörper. J. Reine Angew. Math. **196** (1956), 205–212.

Hasse reformulated the problem of imbedding a given normal extension $\Omega|\Omega_0$ with the Galois group $\mathfrak{g}$ into a normal extension $K|\Omega_0$ with prescribed Galois group $\mathfrak{G}$ for which $\mathfrak{G}|\mathfrak{A}=\mathfrak{g}$ with cyclic $\mathfrak{A}$ as a problem of finding a semi-simple commutative algebra $K|\Omega_0$ with the above properties. In doing so he established, making certain assumptions on the existence of roots of unity in Ω_0, necessary "linking" conditions which relate the quotients $\omega_\chi/\omega_\varphi$ in Ω with $\varphi=\chi^S$, where χ denotes a typical absolutely irreducible character of A, $S\in G$ and ω_χ an element in a factor basis of $K|\Omega$ [H. Hasse, Math. Nachr. **1** (1948), 40–61, 277–283; MR **10**, 426, 503; and see pp. 36-37 of the first paper for a complete formulation of these complicated conditions]. Hasse conjectured loc. cit. that these conditions were also sufficient. However, this is not so, as was pointed out by Faddeev [Dokl. Akad. Nauk SSSR (N.S.) **94** (1954), 1013–1016; MR **15**, 938]. In these papers the author provides a proof of Hasse's conjecture, for arbitrary groups $\mathfrak{g}$ and groups $\mathfrak{A}$ of odd order. Furthermore, it is shown that Faddeev's counter example rests on the impossibility of carrying out simultaneously certain normalizations for general $\mathfrak{A}$ of even order. The author finally derives some additional conditions which suffice to establish the existence of $K|\Omega_0$ for $\mathfrak{A}$ of even order. In the second paper earlier results of H. Richter on imbedding problems over algebraic number fields and their reduction to local imbedding problems [Math. Ann. **112**, 69–84 (1935), 700–726 (1936)] are exploited to exhibit a large class of counter examples. The author also succeeds in avoiding conditions on roots of unity which occurred in the work of Richter.

O. F. G. Schilling (Chicago, Ill.).

Mori, Shinziro. Über Idealtheorie der Multiplikationsringe. J. Sci. Hiroshima Univ. Ser. A. **19** (1956), 429–437.

The author continues his study of the structure of multiplication rings [same J. **16** (1952), 1-11; MR **15**, 676]. A multiplication ring is a commutative ring R that satisfies: If A and B are ideals of R and $A<B$, then there exists an ideal C such that $A=BC$. Let R be a multiplication ring. If $R^2\neq R$, then $\cap_n R^n=0$ and each non-zero ideal of R is a power of R. If $R^2=R$ and each ideal ($\neq 0$ or R) of R has at least one ($\neq R$) prime ideal divisor, then for each ideal A of R, $\cap_n A^n=\cap_i(\cap_n P_i^n)$, where the P_i are the maximal prime ideal divisors of A. *P. F. Conrad.*

Nagahara, Takasi; and Tominaga, Hisao. On Galois theory of division rings. Math. J. Okayama Univ. **6** (1956), 1–21.

The paper refines the outer and almost-outer infinite Galois theories for division rings of Jacobson [Structure of rings, Amer. Math. Soc. Colloq. Publ., V. 37, Providence, R.I., 1956; MR **18**, 373] and Nobusawa [Osaka Math. J. **7** (1955), 1–6; MR **16**, 1084]. The refinement is designed so as to include the Cartan-Jacobson finite Galois theory for division rings, and its main result reads as follows: Let a division ring K be Galois over a subring L; i.e. the fixed subring $J(G(L), K)$ of the group $G(L)$ of all L-automorphisms in K coincides with L. Assume that β) the commuter $V_K(L)$ of L in K is left finite over the center C of K. Then, for $H=V_K(V_K(L))$ we have

$$[K:H]_l=[V_K(L):C]_l.$$

Let $(d_1, \cdots, d_n)$ be a fixed H-left basis of K and set $L_1=L(d_1, \cdots, d_n)$. Assume that γ) L_1 is L-left finite, δ) K is Galois over L_1 and α) H is locally finite over L (i.e. every finite subset of H generates over L a division ring which is L-left finite). Then K is Galois and locally finite over every ring K' between K and L, and every L-isomorphism of K' into K can be extended to an element of $G(L)$. This Galois theorem includes the case where $G=G(L)$ is locally finite-dimensional (which means that $L(S^G)$ is L-left finite for every finite subset S of K and which makes G a topological group in the usual way) and locally compact; in this case the subgroups $G(K')$ of G can be characterized by a certain "regularity" condition. In the course of the proof the paper gives several theorems of independent interest on automorphisms of division rings, including a generalization of the Cartan-Hua-Brauer theorem. *T. Nakayama* (Nagoya).

Nagahara, Takasi. On primitive elements of Galois extensions of division rings. Math. J. Okayama Univ. **6** (1956), 23–28.

Let K be a division ring with center C. For any subring L, L' is the centralizer of L in K; L'' is that of L'. F. Kasch [J. Reine Angew. Math. **189** (1951), 150–159; MR **14**, 239] has proved the following: Theorem A. If (i) K/L is Galois and has finite dimension, and (ii) the center of L' is separable over C, then K is generated over L by at most two conjugates, $K=L(d, u^{-1}du)$, for suitable d, $u \in K$. The present article contains a theorem equivalent to the following generalization of Theorem A: Theorem B. If (i) and (ii) both hold, and if D is any division subring intermediate to K and L satisfying (iii) $x^{-1}Dx=D$ for all nonzero $x \in L'$, then $D=L(d, u^{-1}du)$, for suitable d, $u \in D$. When the Galois group of K/L is locally finite, that is, when for each $a \in K$ the set of images of a by all automorphisms of K/L is finite, the author proves that D/L is simply generated, $D=L(d)$, assuming only (i). [This latter result has been stated in a more general form by N. Nobusawa in Osaka Math. J. **8** (1956), 43–50 [MR **18**, 7]. However, the writer of the review just cited states that the proof of his "Theorem 6" depends upon "Lemma 4" which is false.] Next the author refers to a generalization of the Cartan-Brauer-Hua theorem [see the article reviewed above] to show that (iii) implies that either $D\subsetneq L''$, or that $D\supsetneq L'$. Since the first possibility falls under the locally finite case (the Galois group of L''/L is outer), his main efforts are directed to the case when D contains L_1, the division subring generated by L and L'. In as much as D/L need not be Galois under the assumptions of Theorem B, this theorem properly contains Theorem A. A bit of confusion occurs when the author defines a symbol $R\setminus S$ for two subsets R and S of K which in context reads "the set of elements of R not in S." His terminology: $R\setminus S$ is the complement of R in S. *Carl C. Faith.*

★**Boers, Arie Hendrik. Généralisation de l'associateur.** J. Van Tuyl, Antwerpen–Zaltbommel, 1957. 27 pp.

Seien $a_1, a_2, \cdots$ Elemente eines Ringes R. Verf. definiert $\{a_1, a_2\}=a_1a_2$ und rekursiv

$$\{a_1, a_2, \cdots, a_n\}=\sum_{k=0}^{n-2}(-1)^k\{a_1, a_2, \cdots, a_{k+1}a_{k+2}, \cdots, a_n\}$$

als n-Assoziator, also $\{a_1, a_2, a_3\}=(a_1a_2)a_3-a_1(a_2a_3)$ und $\{a_1, a_2, a_3, a_4\}=\{a_1a_2, a_3, a_4\}-\{a_1, a_2a_3, a_4\}+\{a_1, a_2, a_3a_4\}$ etc. Falls alle n-Assoziatoren verschwinden, heisst R n-assoziativ. Verf. beweist u.a. folgende Struktursätze: Ein 3-assoziativer (d.h. assoziativer) Ring ist n-assoziativ für alle $n\geqq 3$. Umgekehrt gilt, dass eine 4-assoziativer nullteilerfreier Ring assoziativ ist, und dass ein 5-assoziativer nullteilerfreier Ring mit von 2 verschiedener Charakteristik assoziativ ist. Allgemein ist ein n-assoziativer ($n>3$) nullteilerfreier Ring assoziativ, falls seine Charakteristik eine gewisse Nichtteilbarkeitsbedingung erfüllt. Jeder n-Assoziator lässt sich darstellen als eine endliche Summe von Produkten, die zwei Assoziatoren niedrigeren Grades und einen gewissen Binomialkoeffizienten enthalten. — R heisst n-alternativ, wenn jeder n-Assoziator bei Vertauschung von irgend zwei seiner Elemente das Vorzeichen wechselt. Aus dem Verschwinden aller N-Assoziatoren mit zwei gleichen Elementen ($N\geqq 3$) folgt die N-Alternativität. Ist R nullteilerfrei und N-alternativ ($N\geqq 3$), und verschwinden alle $(N+1)$-Assoziatoren, bei denen zwei aufeinanderfolgende Elemente gleich sind, so ist R n-alternativ für alle $n\geqq N$. Ist R nullteilerfrei, $(2n+1)$-assoziativ, und verschwinden alle $(2n+2)$-Assoziatoren, falls zwei konsekutive Elemente übereinstimmen, und ist die Charakteristik von R nicht eine Primzahl $\leqq 2n-2$, so ist R assoziativ. Weitere Sätze beziehen sich auf Bedingungen, unter denen aus dem Verschwinden gewisser k-Assoziatoren auf das Verschwinden aller k'-Assoziatoren mit $k'=k-1$ geschlossen werden kann. — Die Bedingung der Nullteilerfreiheit kann in den genannten Sätzen durch die schwächere Forderung ersetzt werden, dass in R aus $x^2=0$ stets $x=0$ folgt. *R. Moufang.*

Yaqub, Adil. Elementary proofs of the commutativity of p-rings. Amer. Math. Monthly **64** (1957), 253–254.

A proof of number-theoretic nature that every p-ring is commutative, where a p-ring is defined as a ring with unit in which $a^p=a$, $pa=0$ (p prime).

Fleischer, Isidore. Modules of finite rank over Prüfer rings. Ann. of Math. (2) **65** (1957), 250–254.

This paper deals with decomposition theorems for modules over Prüfer rings (integral domains with every finitely generated ideal invertible). The principal tools are the concepts of linearly compact modules (every collection of cosets of submodules has nonvoid intersection provided each finite subcollection has), and linearly precompact modules (every proper homomorph is linearly compact). The author remarks that Dedekind rings and almost maximal valuation rings are linearly precompact as modules over themselves (but the hypotheses of his theorems are not as general as this). The rank of a general module M over an integral domain is defined as the minimum of the ranks of torsion-free modules of which M is a homomorphic image. Some sample propositions are the following. If we consider modules of finite rank over Prüfer rings with linearly compact quotient fields, both the torsion-free modules and the divisible modules are direct sums of modules of rank 1. In fact, over such a ring, every pure submodule of finite rank in a torsion-free module is a direct summand. This eliminates the countability hypothesis in Kaplansky's theorem [Trans. Amer. Math. Soc. **72** (1952), 327–340; MR **13**, 719]. Finally, every submodule of a finitely generated module over a linearly precompact (=almost maximal) valuation ring is a direct sum of modules of rank 1. *D. Zelinsky.*

Vandiver, H. S. Errata: Diophantine equations in certain rings. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 252–253.

Corrections to the article reviewed in MR **18**, 285.

Kasch, Friedrich. Eine Bemerkung über innere Automorphismen. Math. J. Okayama Univ. **6** (1957), 131–133.

L'auteur simplifie des résultats de Nagahara et Tominaga [Proc. Jap. Acad. **31** (1955), 655–658; MR **17**, 578] en démontrant le théorème plus général suivant: soient R un anneau ayant un élément unité, U un sous-anneau simple de R, artinien et contenant l'unité de R. On suppose que U n'est pas contenu dans le centre de R et est infini. Alors il existe $a \in R$ tel qu'il y ait une infinité d'éléments inversibles $u \in U$ pour lesquels les uau^{-1} sont tous distincts. *J. Dieudonné* (Evanston, Ill.).

Szász, F. A. Über die homomorphen Bilder des Ringes der ganzen Zahlen und über eine verwandte Ringfamilie. Monatsh. Math. **61** (1957), 37–41.

Let $R(m, d)$ denote the ring generated by the element a and defined by the relations $ma=0$, $a^2=da$ ($d|m$; m, d non-negative rational integers); and let $R(p)$ denote the ring generated by the elements x, y and defined by the

relations $px=py=yx-x=xy-y=x^2-x=y^3-y=0$ (p a prime). A ring is said to be of property E_1 (respectively E_2) if all its subrings (respectively all its proper finitely generated additive subgroups) have the form Rr ($r \in R$). A ring R has the property E_1 if and only if $R=R(m, 1)$ ($m=0, 1, 2, \cdots$). The class of rings with property E_2 is exhausted by the rings of types $R(0, 1)$, $R(p, p)$, $R(p^m, p)$, $R(2,1)+R(2, 1)$, $\sum_{p \in P} R(p^{k_p}, 1)$ and $R(p)$, where $+$, $\sum$ denote discrete direct sums (in the ring theoretical sense) and P is a subset of the set of all different primes.

A. Kertész (Debrecen).

Hilton, P. J. Note on quasi-Lie rings. Fund. Math. 43 (1956), 230–237.

The author defines a graded Lie ring to be a graded abelian group $L=\sum_{n\geq 0} L_n$ with a multiplication satisfying the identities

1) $xy=(-1)^{(p+1)(q+1)}yx$ $(x \in L_p,\ y \in L_q)$,

2) $$(-1)^{(p+1)r}x(yz)+(-1)^{(q+1)p}y(zx)+(-1)^{(r+1)q}z(xy)=0 \quad (x \in L_p,\ y \in L_q,\ z \in L_r).$$

If one defines $x \circ y=(-1)^p xy$ for $x \in L_p$, one obtains a ring satisfying the usual identities for a graded Lie ring; thus the author's notion is equivalent to the usual one.

One of the main reasons for interest in graded Lie rings is that the homotopy ring of a simply connected space X gives rise to a graded Lie ring. More precisely, let $L_q(X)=\pi_{q+1}(X)$, and use the product in $L(X)=\sum_{q\geq 0}L_q(X)$ determined by the Whitehead product in the homotopy groups. When this is done $L(X)$ is a graded Lie ring.

The axioms for a graded Lie ring have signs which depend only on the parity of the dimension of the elements involved. This leads the author to a generalization of graded Lie rings which he calls quasi-Lie rings. A quasi-Lie ring L is a pair (M, A) where A is a ring such that 1) $aa'=-a'a$, and 2) $a(a'a'')+a'(a''a)+a''(aa')=0$, i.e. A is a Lie ring, and M is an abelian group on which A operates on the left so that $a(a'm)=a'(am)+(aa')m$. Further it is assumed that there is a symmetric bilinear pairing ϕ: $M \times M \to A$ such that

1) $$\phi(m, m')m''+\phi(m', m'')m+\phi(m'', m)m'=0,$$

2) $$a\phi(m, m')=\phi(m, am')+\phi(am, m').$$

Addition and multiplication are defined in $M+A$ by

$$(m_1, a_1)+(m_2, a_2)=(m_1+m_2, a_1+a_2),$$

$$(m_1, a_1)(m_2, a_2)=(a_1m_2+a_2m_1, \phi(m_1, m_2)+a_1a_2).$$

The author proves several theorems about basic products in quasi-Lie rings. We list two of these theorems. Theorem: Suppose L is a quasi-Lie ring, and that $x_1, \cdots, x_r \in M$, $x_{r+1}, \cdots, x_s \in A$. Then the subring generated by $x_1, \cdots, x_s$ is additively generated by the basic products and their squares and cubes formed from the elements $x_1, \cdots, x_s$. Theorem: If L is a free quasi-Lie ring, then it is additively the direct sum of 1) a free abelian group, 2) a free mod 2 module, and 3) a free mod 3 module. Actually the author specifies bases for the three terms in the preceding decomposition in terms of basic products of a set of generators for L.

J. C. Moore.

Szele, T.; and Fuchs, L. On Artinian rings. Acta Sci. Math. Szeged 17 (1956), 30–40.

The author studies the structure of rings whose left ideals satisfy the minimum condition (i.e., Artinian rings). First, the structure of rings expressible as a direct sum of a finite number of minimal left ideals is described. In particular, it is shown that such rings are characterized by a finite number of division rings and natural numbers. In the remainder of the paper, a theorem of L. Fuchs [Publ. Math. Debrecen 4 (1956), 488–508; MR 18, 188] characterizing the additive group of an Artinian ring is exploited. For example, it is shown that the following statements about an Artinian ring A are equivalent. (1) The additive group of A contains no subgroup of type p^∞. (2) A can be embedded in an Artinian ring with unity element. (3) The left ideals of A satisfy the maximum condition. In addition, a necessary and sufficient condition that the radical of an Artinian ring be an Artinian ring is given.

M. Henriksen (Princeton, N.J.).

Iséki, Kiyoshi; and Miyanaga, Yasue. On a radical in a semiring. Proc. Japan Acad. 32 (1956), 562–563.

This note continues the study of ideals in semirings begun by Słowikowski and Zawadowski [Fund. Math. 42 (1955), 215–231; MR 18, 223] and continued by the authors [Proc. Japan Acad. 32 (1956), 325–328; MR 18, 223]. Terminology and notation are as in the first review cited. Let A be a not necessarily commutative semiring with unit and zero. An element $a \in A$ is a left zero divisor if $ab=0$ for some $b\neq 0$. A left ideal $\mathfrak{A}$ consisting entirely of left zero divisors is called a left zero divisor ideal. Following L. Fuchs [Acta Sci. Math. Szeged 16 (1955), 43–53; MR 17, 8], the authors define a left zeroid ideal as a left ideal $\mathfrak{A}$ such that $\mathfrak{A}+\mathfrak{B}$ is a left zero divisor ideal for every left zero divisor ideal $\mathfrak{B}$. The join of all left zeroid ideals is the left radical $\mathfrak{R}^{(l)}$ of A. The right radical $\mathfrak{R}^{(r)}$ is defined similarly, and the radical $\mathfrak{R}$ of A is defined as $\mathfrak{R}^{(l)} \frown \mathfrak{R}^{(r)}$. Theorem 1: $\mathfrak{R}^{(l)}$ is the intersection of all maximal left zero divisor ideals in A. (Hence $\mathfrak{R}$ has an obvious characterization as well.) Theorem 2: Every maximal left zero divisor ideal is prime (in the sense of the paper reviewed below).

E. Hewitt.

Iséki, Kiyoshi. Ideal theory of semiring. Proc. Japan Acad. 32 (1956), 554–559.

Notation and terminology are as in the preceding review. Let A be a semiring, with or without a unit. An ideal $\mathfrak{J} \subset A$ is said to be prime if $\mathfrak{A}\mathfrak{B} \subset I$ for ideals $\mathfrak{A}$ and $\mathfrak{B}$ implies $\mathfrak{A} \subset \mathfrak{J}$ or $\mathfrak{B} \subset \mathfrak{J}$. An ideal $\mathfrak{J}$ is said to be irreducible (strongly irreducible) if $\mathfrak{A} \frown \mathfrak{B}=\mathfrak{J}$ ($\mathfrak{A} \frown \mathfrak{B} \subset \mathfrak{J}$) for ideals $\mathfrak{A}$ and $\mathfrak{B}$ implies that $\mathfrak{A} \subset \mathfrak{J}$ or $\mathfrak{B} \subset \mathfrak{J}$. (L. Fuchs [Comment. Math. Helv. 23 (1949), 334–341; MR 11, 310] has already introduced the notion of strongly irreducible ideals, calling them primitive.) For $a \in A$, let (a) be the principal ideal generated by a. Theorem: An ideal $\mathfrak{J}$ is prime if and only if $a, b \in \mathfrak{J}'$ implies $axb \in \mathfrak{J}'$ for some $x \in A$. Theorem: An ideal $\mathfrak{J}$ is strongly irreducible if and only if $a, b \in \mathfrak{J}'$ implies that $(a) \frown (b) \frown I'$ is non-void. Theorem: Every ideal in A is the intersection of the irreducible ideals that contain it. Theorem: Let A be commutative and have a unit. Let $\mathscr{S}$ be the set of all strongly irreducible ideals in A, with the topology in which the sets

$$\mathscr{T}_x=\{\mathfrak{J} : \mathfrak{J} \in \mathscr{S},\ x \notin \mathfrak{J}\}$$

are an open basis. Then $\mathscr{S}$ is a compact T_0-space. A number of similar results are also obtained. For an analogous study of maximal ideals, see the author and Miyanaga [Proc. Japan Acad. 32 (1956), 325–328; MR 18, 223].

E. Hewitt (Seattle, Wash.).

Lech, Christer. On the associativity formula for multiplicities. Ark. Mat. **3** (1957), 301–314.

Etant donnés un anneau local Q et un idéal $\mathfrak{q}$ primaire pour l'idéal maximal de Q, nous noterons $e(\mathfrak{q})$ sa multiplicité, et $l(\mathfrak{q})$ sa longueur (c.à.d. celle de $Q/\mathfrak{q}$ au sens du théorème de Jordan-Hölder). Soient r la dimension de Q, $(x_1, \cdots, x_r)$ un système de paramètres de Q, $\mathfrak{q}$ l'idéal $(x_1, \cdots, x_r)$ et $\mathfrak{b}$ l'idéal $(x_{m+1}, \cdots, x_r)$ $(n \leqq r)$. On a alors

$$e(\mathfrak{q}) = \sum e((\mathfrak{q}+\mathfrak{p})/\mathfrak{p})e(\mathfrak{b}Q_{\mathfrak{p}}),$$

où la sommation est étendue aux idéaux premiers isolés $\mathfrak{p}$ de $\mathfrak{b}$ pour lesquels $\dim(Q/\mathfrak{p})+\dim(Q_{\mathfrak{p}})=\dim(Q)=r$; ce résultat généralise la formule d'associativité de Chevalley.

La démonstration est basée sur l'intéressant lemme suivant: lorsque les exposants $s(i)$ tendent vers l'infini, le quotient $l(x_1^{s(1)}, \cdots, x_r^{s(r)})/s(1)\cdots s(r)$ tend vers

$$e(x_1, \cdots, x_r)$$

(où $(x_1, \cdots, x_r)$ est un système de paramètres de Q); la démonstration de ce lemme est de nature asymptotique. On démontre ensuite la formule d'associativité dans le cas $m=r$ (c'est à dire $\mathfrak{b}=(0)$) en utilisant le lemme d'Artin-Rees (dans une première version, antérieure au lemme d'Artin-Rees, l'auteur passait d'abord par le cas $m=r=1$). La formule générale se déduit alors, par récurrence sur r, du premier lemme et d'une formule faisant intervenir les chaines d'idéaux premiers „adaptées" au système de paramètres $(x_1, \cdots, x_r)$.

Applications aux idéaux analytiquement disjoints, et à la démonstration des formules $e(x_1^{s(1)}, \cdots, x_r^{s(r)}) = s(1)\cdots s(r)\ e(x_1, \cdots, x_r)$ et $e(xx')=e(x)+e(x')$, où x et x' désignent des paramètres d'un anneau local de dimension 1.

P. Samuel (Clermont-Ferrand).

Guérindon, Jean. Recherche d'invariants de modules usuels. C. R. Acad. Sci. Paris **244** (1957), 1863–1866.

Des relations entre la théorie de l'irréductibilité et la théorie multiplicative étudiées antérieurement [mêmes C. R. **242** (1956), 2693–2695; **243** (1956), 936–939; MR **18**, 8, 277] sont établies en considérant une classe usuelle de modules, ou modules de Krull, et cela permet de trouver des systèmes d'invariants tous finis. (Author's summary.)

D. C. Murdoch (Vancouver, B.C.).

★**Barbilian, D. Teoria aritmetica a idealelor (în inele necomutative). [Arithmetic theory of ideals (in noncommutative rings).]** Editura Academiei Republicii Populare Romîne, 1956. 379 pp. Lei 15,35.

Mitas, Günter. Über Primpolynomzerlegung in endlich vielen Schritten. J. Reine Angew. Math. **197** (1957), 76–81.

The author is concerned with fields Ω having the property (A): Every polynomial in $\Omega[x]$ can be split into its irreducible factors in finitely many steps. Theorem 1. Ω has (A) if Ω has both of the following properties: (A1) every polynomial relatively prime to its derivative can be split as above; and (A2) if the characteristic $p\neq 0$, then every polynomial of the form x^p-a may be split as above. Theorem 2. If Ω has (A1), then so has $\Omega(b)$ whenever b is transcendental over Ω or separably algebraic over Ω or (for $p\neq 0$) $b^p \in \Omega$. Theorem 3. If Ω has (A2), then so has $\Omega(b)$ in the first two cases mentioned in theorem 2, and, provided $p\neq 0$ and $[\Omega:\Omega^p]=p$, also in the third case. That the proviso is not superfluous is shown by an example due to M. Kneser [Math. Z. **57** (1953), 238–240; MR **14**, 613]. The proof of theorem 2 makes use of the paper reviewed below.

E. R. Kolchin (New York, N.Y.).

See also: Hodges, p. 6; Mitas, p. 11; Nakano, p. 18; Gröbner, p. 63; Northcott, p. 63.

Algebras

Mitas, Günter. Bemerkungen zur Algebrentheorie. J. Reine Angew. Math. **197** (1957), 68–75.

The author extends to algebras of finite rank well-known theorems about finite algebraic field extensions. Because he wishes subsequently to apply his results to questions concerning constructive methods, he is careful to make his proofs constructive and independent of the results to be developed later. He uses the language of matrices, but the reviewer found it helped his own understanding to translate the statements into more intrinsic language. Let A be an algebra of finite rank over a field Ω. For any $a \in A$ let $S(a)$ denote the trace of the image of a in the regular representation of A. Definition: A is regular-separable if the bilinear form $(a, b)\to S(ab)$ on A is nondegenerate. It is well-known that this notion is, if the field characteristic is 0 or if A is commutative, equivalent to the separability of A. Also, this notion is readily extended to finite-dimensional algebras over any commutative ring with unity. Theorem 1.1: If $f \in \Omega[x]$, then $\Omega[x]/(f)$ is regular-separable if and only if f is relatively prime to f'. Let B be an algebra over Ω of rank ν with unity, and let A_0 be an algebra of rank n over the center C of B; the tensor product $A_0 \otimes_C B$ is an algebra over Ω of rank $n\nu$. Theorem 2.3: $A_0 \otimes_C B$ is regular-separable if and only if A_0 and B are. Now, let $B=\Omega[y]/(\varphi)$, where $\varphi \in \Omega[y]$; then $b=y \pmod{\varphi}$ is a primitive element of B over Ω in the sense that $B=\sum \Omega b^j$. Similarly, let $A=B[x]/(f)$, where $f \in B[x]$, so that $a=x \pmod{f}$ is a primitive element of A over B. Theorem 3.3: If B is regular-separable and Ω is infinite, there exists a $\delta \in \mathfrak{O}$ such that $a+\delta b$ is a primitive element of A over Ω.

E. R. Kolchin (New York, N.Y.).

Hoehnke, Hans-Jürgen. Über die definierenden Gleichungen für Matrizeneinheiten in primären Ringen. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **6** (1956/57), 1–4.

Let $\mathfrak{D}$ be a completely primary ring and let $\mathfrak{A}$ be the primary ring of all r by r matrices with elements in $\mathfrak{D}$. Let e_{ij} $(i, j=1, \cdots, r)$ be the usual matric units of $\mathfrak{A}$ and let $\mathfrak{G}$ be the multiplicative semigroup of all elements de_{ij} with $d \in \mathfrak{D}$ and $i, j=1, \cdots, r$. If $\mathfrak{H}$ is a multiplicative semigroup such that $\mathfrak{G}\subseteq\mathfrak{H}\subseteq\mathfrak{A}$, while θ is a semigroup homomorphism of $\mathfrak{H}$ into $\mathfrak{A}$ such that $0^\theta=0$ and $\mathfrak{G}^\theta\neq 0$, it is shown that there is a nonsingular element $t \in \mathfrak{A}$ such that $x^\theta=t[x_{ij}^\varphi]t^{-1}$ for all $x \in \mathfrak{H}$, where φ is an endomorphism of the multiplicative semigroup of $\mathfrak{D}$ which satisfies $1^\varphi=1$ and also $0^\varphi=0$ if $r\geqq 2$. When $r\geqq 2$ and $\mathfrak{H}$ contains all elements of the form $e_{11}+de_{12}$, $de_{11}+e_{21}$ with $d \in \mathfrak{D}$, then the author shows that φ is a ring endomorphism of $\mathfrak{D}$. The proofs rest on the observation that every nontrivial solution u_{ij} $(i, j=1, \cdots, r)$ of the equations $u_{ij}u_{kl}=\delta_{jk}u_{il}$ $(i, j, k, l=1, \cdots, r;\ \delta_{jk}=$ Kronecker delta) in $\mathfrak{A}$ automatically satisfies $\sum_i u_{ii}=1$ and hence also $u_{ij}=te_{ij}t^{-1}$ for some nonsingular $t \in \mathfrak{A}$. These results extend those of Mal'cev [Dokl. Akad. Nauk SSSR (N.S.) **90** (1953), 333–335; MR **14**, 1057] and of Halezov [ibid. **96** (1954), 245–248; MR **16**, 333] who studied the case in which $\mathfrak{D}$ is a field and $\mathfrak{H}=\mathfrak{H}(m)$ consists of all matrices of $\mathfrak{A}$ whose rank is at most m.

M. F. Smiley.

See also: Kasch, p. 9.

Groups and Generalizations

Chen, Kuo-Tsai. Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula. Ann. of Math. (2) **65** (1957), 163–178.

To each path $\alpha:\langle\alpha_1(t), \cdots, \alpha_m(t)\rangle$, $a\leqq t\leqq b$ in Euclidean m-dimensional space R^m is associated the formal power series $\theta(\alpha)=1+\sum_{p=1}^{\infty} \sum I_{t_1\cdots t_p}(\alpha)X_{t_1}\cdots X_{t_p}$ in the non-commutative indeterminates $X_1, \cdots, X_m$, where

$$I_{t_1}\cdots{}_{t_p}(\alpha)=\int_a^b\int_a^{t_p}\cdots\int_a^{t_2}d\alpha_{t_1}(t_1)\cdots d\alpha_{t_{p-1}}(t_{p-1})d\alpha_{t_p}(t_p)$$

[cf. Chen., Proc. London Math. Soc. (3) **4** (1954), 502–512; MR **17**, 394]. If a linear transformation is applied to R^m the coefficients of the terms of pth order transform as pth order contravariant tensors. From these tensors various invariants of α may be derived. For instance the coefficients of the 2d order terms form a matrix whose characteristic polynomial is an invariant. To each path α is assigned a 'lower central class', — the least positive integer p for which there is a non-vanishing coefficient of a pth order term. The main theorem is that $\log\theta(\alpha)$ is a Lie element, i.e. the pth order terms can be expressed for each $p\geqq1$ as a linear combination of bracket products. Here $\log(1+u)$ means, as usual, the formal power series $u-u^2/2+u^3/3-\cdots$. This result is a generalization of the Baker-Hausdorff formula since

$$\log\theta(\alpha\beta)=\log(\theta(\alpha)\theta(\beta))=\log(\exp X\exp Y))$$

if α and β are unit horizontal and vertical segments in the xy-plane. In the last section it is indicated how R^m can be replaced by an arbitrary m-dimensional differentiable manifold. *R. H. Fox* (Princeton, N.J.).

Kirschmer, G. Über eine mit den Pythagoräischen Zahlen zusammenhängende Gruppe. Elem. Math. **12** (1957), 49–56.

Die Elemente der Gruppe sind die Zahlentripel $A[a_1;\ a_2;\ a_3]$, $B[b_1;\ b_2;\ b_3]$, $\cdots$, mit $a_1^2=a_2^2+a_3^2$, $b_1^2=b_2^2+b_3^2$, $\cdots$, $a_3b_3\neq0$. Die Gruppenmultiplikation lautet

$$AB=[a_1b_1+a_2b_2;\ a_1b_2+a_2b_1;\ a_3b_3].$$

Diese Gruppe ist vielseitig deutbar und bietet verschiedene Anwendungsmöglichkeiten. *Aus der Einleitung.*

Lyahovickiĭ, V. N. On the question of decomposability of a group into differently constituted nilpotent products. Mat. Sb. N.S. **40(82)** (1956), 401–414. (Russian)

[For definitions and terminology, see Golovin, Mat. Sb. N.S. **27(69)** (1950), 427–454; **28(70)** (1951), 431–444, 445–452; MR **12**, 672; **13**, 105.] The author forms the metabelian product of two abelian groups, A and B, which are direct sums of cyclic groups, and then constructs the intersections of the center of each with the centralizer in the metabelian product of the other factor. If A and B are periodic, he shows that any prime number which appears in the order of an element of one of these intersections cannot so occur in the other. Now let A and B be two groups that have factor-commutator groups which are direct sums of cyclic groups. The author shows that characterizing conditions that the metabelian product of A and B which is not at the same time the direct product of A and B split into a direct product are that A and B so split into $A_1\times A_2$ and $B_1\times B_2$ and that the metabelian commutator subgroups of the pairs A_1, B_2 and A_2, B_1 are trivial. In particular, a finite p-group which can be represented as a metabelian product is not a direct product. Since a special case only has been settled here, the general question of the title remains open. *F. Haimo* (St. Louis, Mo.).

Baer, Reinhold. Lokal Noethersche Gruppen. Math. Z. **66** (1957), 341–363.

A group is Noetherian if every subgroup is finitely generated, locally Noetherian if every finitely generated subgroup is Noetherian, and almost solvable if every homomorphic image (except 1) has a normal subgroup $N\neq1$ with finite commutator subgroup. An extension of a locally Noetherian group G by a finite group is locally Noetherian, but an extension of G by an infinite cyclic group need not be Noetherian. These facts lead the author to a search for conditions under which an extension of a locally Noetherian group by a locally Noetherian group is again a locally Noetherian group. It is shown that if N and D are normal in $G=NU$, $D\subset N\cap U$, and N, U, and G/D are locally Noetherian, then G is locally Noetherian. Again, if N is a locally Noetherian normal subgroup of G such that (i) every extension of N in G by an infinite cyclic group is locally Noetherian, and (ii) every finitely generated subgroup of G/N is Noetherian and almost solvable, then G is locally Noetherian. The first of these theorems implies that there is a largest locally Noetherian subinvariant subgroup of any group G, where the term subinvariant means transfinitely subinvariant. Noetherian almost solvable groups are characterized in several ways; for example, they coincide with finite extensions of Noetherian solvable groups (G is solvable if every factor group $G/N\neq1$ has an abelian normal subgroup not 1).

The paper also contains a more general study of classes Σ of groups which are hereditary (closed under taking of subgroups and homomorphic images), and with various further properties, for example, (P) if M and N are normal in G and both in Σ, then $MN\in\Sigma$, or (T) if $\{H_\alpha\}$ is a well-ordered increasing sequence of groups in Σ, then $\bigcup H_\alpha\in\Sigma$. With each hereditary class Σ, there is defined a hereditary class $L(\Sigma)$, where $G\in L(\Sigma)$ if and only if every finitely generated subgroup H of G is in Σ. Three sample results follow. (A) If Σ is the class of Noetherian almost solvable groups, then Σ is hereditary and satisfies (P) and the condition: if $N\in\Sigma$ and $G/N\in\Sigma$, then $G\in\Sigma$. (B) If Σ is a hereditary class satisfying (P) and (T), then for any group G, the subgroup S generated by all subinvariant subgroups H of G such that $H\in\Sigma$ is characteristic and in Σ. (C) If Σ is a hereditary class of Noetherian groups, then Σ has property (P) if and only if (*) if N_α is in $L(\Sigma)$ and is normal in G, then the product of the N_α is in $L(\Sigma)$. *W. R. Scott* (Lawrence, Kansas).

Fuchs, L.; Kertész, A.; and Szele, T. On abelian groups in which every homomorphic image can be imbedded. Acta Math. Acad. Sci. Hungar. **7** (1956), 467–475. (Russian summary)

A group G has property Q if every homomorphic image of G can be isomorphically imbedded in G. The authors show that an abelian p-group has Q if and only if it has a direct summand which is the direct sum of $\mathfrak{m}$ copies of groups of type p^∞, where $\mathfrak{m}=\min_n \operatorname{rank}(p^nG)$, the final rank of G. If abelian G is of infinite torsion-free rank $\mathfrak{r}$, if $\mathfrak{p}_i$ is the final rank of the p_ith component of the torsion subgroup of G, then Q holds if and only if $\mathfrak{r}\leqq$each $\mathfrak{p}_i$ and G has a direct summand which is the direct sum over all primes of direct sums of groups of type p_i^∞ and of the

direct sum of $\mathfrak{r}$ copies of the additive group of rationals. If, however, $\mathfrak{r}<\aleph_0$, then G has Q if and only if G splits into a direct sum of a torsion-free group F of rank $\mathfrak{r}$ and a direct sum of p_i-groups of infinite final rank $\mathfrak{p}_i$, each with the property Q, where F has a further decomposition described in the paper. Reference is made to the authors' paper [Acta Sci. Math. Szeged **16** (1955), 77–88; MR **16**, 1086] where the dual problem is discussed. *F. Haimo.*

Gol'dina, N. P. Free nilpotent groups. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 528–530. (Russian)

Some properties of the groups of the title are announced without proofs. For instance, in a free n-step nilpotent group G, the basic commutators of weight t generate a free nilpotent group of class $[(n+t-1)/t]$ and constitute its free generators. In the tth member of the lower central series of G, a set which is linearly independent modulo the $t+1$-st member of this series likewise so generates and constitutes. There is an essential converse to the last statement. Reference is made to Mal'cev [Mat. Sb. N.S. **37(79)** (1955), 567–572; MR **17**, 345]. *F. Haimo.*

Trofimov, P. I. The type of finite nilpotent groups with three classes of noninvariant conjugate subgroups. Tomskiĭ Gos. Univ. Uč. Zap. Mat. Meh. **25** (1955), 43–44. (Russian)

Let Γ be nilpotent and have exactly three classes of noninvariant conjugate subgroups. Then its order τ is divisible by exactly two primes p, g: $\tau=p^\alpha q^\beta$. The author shows that Γ is the direct product of Π, Λ, where $o(\Pi)=p^\alpha$, $o(\Lambda)=q^2$; Λ is cyclic; Π is as defined in an earlier review [same Zap. **25** (1955), 40–42; MR **18**, 377]. The three classes consist of groups of orders p, pq, pq^2 respectively. *J. L. Brenner* (Palo Alto, Calif.).

Čarin, V. S. On locally solvable groups of finite rank. Mat. Sb. N.S. **41(83)** (1957), 37–48. (Russian)

A group of finite rank [Mal'cev's finite special rank, Mat. Sb. N.S. **22(64)** (1948), 351–352; MR **9**, 493] which is locally solvable and torsion-free is shown to be a solvable group. The second section studies one class of solvable groups: in a group having a rational series of finite length, what properties of such series are invariant, depending only on the group? *R. A. Good.*

Černihin, S. A. Factorization of finite groups. Mat. Sb. N.S. **39(81)** (1956), 465–490. (Russian)

Details are given for the results announced in two previous papers [Dokl. Akad. Nauk SSSR (N.S.) **97** (1954), 977–980; **103** (1955), 377–378; MR **16**, 331; **17**, 235]. *R. A. Good* (College Park, Md.).

McLain, D. H. The existence of subgroups of given order in finite groups. Proc. Cambridge Philos. Soc. **53** (1957), 278–285.

For a group G and a subgroup H the number n is called a possible order between G and H if it divides the order $o(G)$ of G and is divisible by the order $o(H)$ of H. The author first derives the result (due to Zappa) that every subgroup H contains subgroups for every divisor of $o(H)$ only if G is supersoluble. Next follows a discussion of groups with the properly that every subgroup K of G is contained in a subgroup of every possible order. Such groups must also be supersoluble, but the question of their exact structure is still open. The author gives various equivalent definitions and reduces the problem to the case where $o(G)$ is divisible only by two primes. *O. Ore.*

Jaffard, P. Sur les groupes réticulés associés à un groupe ordonné. Publ. Sci. Univ. Alger. Sér. A. **2** (1955), 173–203 (1957).

Eine geordnete Gruppe G — alle Gruppen seien hier abelsch — ist bekanntlich genau dann zu einer Verbandsgruppe erweiterbar, wenn sie semi-abgeschlossen (s-abgeschlossen) ist: $x^n\geqq 1\rightarrow x\geqq 1$ für alle $x\in G$. Die Verbandsgruppen, $\Lambda\geqq G$, die keine Unterverbandsgruppe Λ' mit $\Lambda\supset\Lambda'\leqq G$ besitzen, werden vom Verfasser „zu G gehörige Lorenzensche Gruppen" genannt. Sie sind konstruierbar, indem man von den r-Idealsystemen von G ausgeht, für die G r-abgeschlossen ist [vgl. Lorenzen, Math. Z. **45** (1939), 533–553; MR **1**, 101]. Über diese Gruppen beweist Verfasser u.a.: (1) Eine Verbandsgruppe ist stets direkter Faktor jeder ihrer Lorenzenschen Gruppen. Jede s-abgeschlossene Gruppe besitzt charakteristische Lorenzensche Gruppen, d.h. solche, die keinen Verbandsgruppenhomomorphismus, der G fest lässt, gestatten. (3) Bilden die endlichen r-Ideale von G schon eine Gruppe, so entsprechen die t-Ideale $\mathfrak{Q}$ von Λ eineindeutig den r_s-Idealen $\mathfrak{Q}\cap G$ von G. Hierbei entsprechen einander auch die Primidealen. Bilden die endlichen r-Idealen dagegen nur eine Halbgruppe, so kann für t-Primideale $\mathfrak{P}_1\subset\mathfrak{P}_2$ (ebenso für $\mathfrak{P}_1\not\subseteq\mathfrak{P}_2$ und $\mathfrak{P}_2\not\subseteq\mathfrak{P}_1$) doch $\mathfrak{P}_1\cap G=\mathfrak{P}_2\cap G$ gelten. Für ein minimales t-Primideal $\mathfrak{P}$ braucht $\mathfrak{P}\cap G$ nicht minimal zu sein. *P. Lorenzen.*

★ **Gelfand, I. M.; und Neumark, M. A. Unitäre Darstellungen der klassischen Gruppen.** Akademie-Verlag, Berlin, 1957. XL+333 pp. DM 36.00.

A translation, prepared by the Forschungsinstitut für Mathematik of the Deutsche Akademie der Wissenschaften zu Berlin, from the Russian of the book reviewed in MR **13**, 722. The present volume is Band VI of the II. Abteilung (Mathematische Monographien) of the Mathematische Lehrbücher und Monographien of the above Forschungsinstitut.

Kemperman, J. H. B. On complexes in a semigroup. Nederl. Akad. Wetensch. Proc. Ser. A. **59** (1956), 247–254.

Generalizing a theorem of P. Scherk and the reviewer, the author proves a number of "density" theorems for semigroups of which the following is representative. Let A and B be subsets of an additively written semigroup G, and let $[A]$ and $[B]$ denote the order of these sets. An element c in G is called invertible when G can be embedded in a semigroup G' containing o and $-c$. Suppose that some invertible element c in $A+B$ admits at most m different representations $c=a+b$. Then the number of distinct elements in $A+B$ is at least $[A]+[B]-m$. *L. Moser* (Edmonton, Alta.).

Dickinson, D. J. On Fletcher's paper „Campanological groups". Amer. Math. Monthly **64** (1957), 331–332.

The author draws attention to the fact that the "tedious process of enumeration" of cosets referred to by Fletcher [same Monthly **63** (1956), 619–626; MR **18**, 377] can be avoided by using a theorem of the reviewer [Proc. Cambridge Philos. Soc. **44** (1948), 17–25; MR **9**, 267] and gives a proof for the case under consideration. *R. A. Rankin* (Glasgow).

See also: Yacoub, p. 5; Hoehnke, p. 11; Newman, p. 17.

Homological Algebra

★ **Eckmann, Beno. Zur Cohomologie-theorie von Räumen und Gruppen.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 170–177. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

An expository article dealing mostly with the author's own results [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 35–42; Ann. of Math. (2) **58** (1953), 481–493; MR **15**, 459, 397]. *S. Eilenberg* (New York, N.Y.).

Auslander, Maurice. On the dimension of modules and algebras. VI. Comparison of global and algebra dimension. Nagoya Math. J. **11** (1957), 61–65.

Let Λ be a semi-primary algebra over a field K and with radical N. It is shown that if $\dim \Lambda<\infty$ ($\dim \Lambda$ is the cohomological dimension of the K-algebra Λ) and if $(\Lambda/N{:}K)<\infty$, then $\dim \Lambda = \text{gl. dim } \Lambda$ (gl. dim Λ is the global dimension of the ring Λ). The proof is based on the following additivity theorem. Let Λ_i be K-algebras with radical N_i $(i=1, 2)$; if gl. $\dim(\Lambda_1\otimes_K\Lambda_2)<\infty$, then

$$\text{gl. dim}(\Lambda_1\otimes_K\Lambda_2)=\text{gl. dim } \Lambda_1+\text{gl. dim } \Lambda_2.$$

S. Eilenberg (New York, N.Y.).

See also: Adem, p. 50; Eilenberg and Ganea, p. 52.

THEORY OF NUMBERS

General Theory of Numbers

Golubev, V. A. Sur un curieux résultat arithmétique. Mathesis **66** (1957), 25–28.

Discussion of the following already known fact: étant donné un nombre quelconque de quatre chiffres non tous égaux, on forme avec ceux-ci le plus grand nombre M et le plus petit nombre m possibles; on détermine la différence $r=M-m$ que l'on complète par des zéros si elle présente moins de quatre chiffres et l'on fait sur r les mêmes opérations; en répétant ces opérations on arrive finalement à la différence 6174 qui se réproduit indéfiniment. *From the introduction.*

Kluge, Théodore. Sur la solution générale d'un problème de Fermat. Mathesis **66** (1957), 22–24.

A method for the factorization of large numbers.

Robinson, Raphael M. Factors of Fermat numbers. Math. Tables Aids Comput. **11** (1957), 21–22.

Fourteen new factors of Fermat numbers $2^{2^n}+1$ as discovered by the SWAC are listed. They are all "small", that is, they are of the form $k\cdot 2^{n+2}+1$ for small values of k, less than 30 with one exception, 1575. The values of n involved are
39, 55, 63, 117, 125, 144, 150, 207, 226, 228, 268, 284, 316, 452.
A list of 32 large primes of the form $k\cdot 2^h+1$ for $k=3, 5, 7$ and $2<h<1024$ is also given. *D. H. Lehmer.*

Roberts, J. B. A curious sequence of signs. Amer. Math. Monthly **64** (1957), 317–322.

Let $b_j(n)$ be the coefficient of b^j in the expansion of n to base b and let $v_b(n)=\sum_j b_j$. The principal function studied here is

$$u_b(n)=(-1)^{v_b(n)}\prod_j\binom{b-1}{b_j(n-1)}.$$

The case $b=2$ yields the sequence of signs referred to in the title and is used to define a set of functions which form a lacunary subsequence of the Walsh function. General properties of $u_b(n)$ include

$$u_b(n+mb^k)=u_b(n)u_b(m+1)\quad(1\leq n\leq b^k),$$
$$\sum_{n=1}^{b^k} u_b(n)(x+n)^m=0\quad(0\leq m<k(b-1)).$$

Applications of the last result to the Tarry-Escott problem are discussed. *L. Moser* (Edmonton, Alta.).

Kasch, Friedrich. Wesentliche Komponenten bei Gitterpunktmengen. J. Reine Angew. Math. **197** (1957), 208–215.

An r-dimensional non-negative lattice point is an ordered set of r nonnegative rational integers. The sum of two such lattice points is computed by the rule of vector addition, and the lattice point $(0, 0, \cdots, 0)$ is denoted by 0. Let A and B denote sets of non-negative lattice points in r-dimensional space, and $A+B$ the set of all points $a+b$, $a\in A$, $b\in B$. If $m=(m_1, \cdots, m_r)$ and $n=(n_1, \cdots, n_r)$ are two points, and if $m_i\leq n_i$ $(i=1, 2, \cdots, r)$, we write $m\leq n$. With this convention we let $A(n)$ stand for the number of those points a of A for which $a\leq n$. We set $\alpha=\inf_{n\in N} A(n)/N(n)$, where N is the set of all non-negative lattice points other than 0; thus α is the density of A. The author proves that if B is a set which contains 0 and has positive density, then B constitutes a basis of finite order for the points of N. This result is well-known for $r=1$, and the general theorem is derived from the linear case. Let $l(m)$ denote the smallest l such that $m=b_1+b_2+\cdots+b_l$, $b_i\in B$ $(i=1, 2, \cdots, l)$, $m\in N$; then the theorem states that $\sup_{m\in N} l(m)$ is finite. Let λ, the average order of B, be given by

$$\lambda=\sup_{n\in N} N^{-1}(n)\textstyle\sum_{m\leq n} l(m).$$

The author proves two main results, Theorems 2 and 4. The first states that if B is any basis containing 0, with finite average order λ, and A is any set with density α, then there exists a constant $c>0$ such that γ, the density of the set $A+B$, satisfies

$$\gamma\geq\alpha\left(1+c\frac{1-\alpha}{\lambda}\right);$$

moreover, c may be taken as $\{3(r+1)\}^{-r}$. The result for $r=1$ is due to Erdös [Acta Arithmetica **1** (1936), 197–200] who proved it with $c=\frac{1}{2}$; the proof is based on the author's previous paper [Math. Z. **64** (1956), 243–257; MR **17**, 1060]. A corollary of Theorem 2 (Theorem 3) is a generalisation to r dimensions of Schnirelmann's inequality for densities of linear sets [Über einige neuere Fortschritte der additiven Zahlentheorie, Cambridge, 1937]. The author observes that this generalization is not sharp, although anything as good as Mann's Theorem in one dimension has been proved false for higher dimensions. Section I concludes with some remarks about corresponding results for asymptotic densities. In Section II the author proves Theorem 4, his second principal result, which is concerned with plane sets only and gives a generalisation of Schnirelmann's inequality sharper than

Theorem 3 for $r=2$. It is shown that if A and B are two plane sets with densities α and β respectively, and if $0 \in B$, then the density γ of the set $A+B$ satisfies

$$\gamma \geqq \alpha + \frac{\alpha}{2}(1-\alpha)\beta.$$

The author is unable to extend this result to higher dimensions, because the proof depends too much on properties of linear sets, but he conjectures that in r-dimensional space the result

$$\gamma \geqq \alpha + \left(\frac{\alpha}{r}\right)^{r-1}(1-\alpha)\beta$$

is true. The question of the sharpest possible generalisation remains open. *H. Halberstam* (London).

Czarnota, A. The necessary and sufficient conditions for the modules of the congruence $\sum_{r=1}^{n-1} r^{n-1} \equiv -1 \pmod{n}$. Prace Mat. 2 (1956), 172–178. (Polish. Russian and English summaries)

Der Verf. knüpft an die Arbeit von S. Sispánov [Bol. Mat. 14 (1941), 99–106; MR 3, 66] an und beweist folgenden Satz; Die notwendige und hinreichende Bedingung für die Gültigkeit der Kongruenz $\sum_{r=1}^{n-1} r^{n-1} \equiv -1 \pmod{n}$ ($n \geqq 3$, ganz) ist, dass die Zahl n von der Gestalt $n = p_1 p_2 \cdots p_s$ ($s \geqq 1$) sei, wo p_i verschiedene ungerade Primzahlen sind, für die $(p_i - 1) | (n-1)$ ($i = 1, \cdots, s$) und $\sum_{i=1}^{s} n/p_i \equiv 1 \pmod{n}$ gilt. Es ist aber nicht sicher, ob ausser den ungeraden Primzahlen überhaupt noch weitere n existieren, die diese Bedingungen erfüllen. Wenn es nämlich so wäre, dann müsste — wie der Verf. beweist — eine solche Zahl n grösser als das Produkt von allen ungeraden Primzahlen von 3 bis 29, d.h. grösser als 3 234 846 615 sein. *V. Knichal* (Prag).

Ginatempo, Nicola. Sulla risoluzione in numeri interi della equazione $x^4 - 8y^4 - 8y^2 z = z^2$ Atti Soc. Peloritana Sci. Fis. Mat. Nat. 2 (1955–56), 13–25.

A proof at great length that the complete solution of the title equation in integers is equivalent to the complete solution in rationals h, ω of $2h^3 + 4h = \omega^2$, and also to the finding of all integral Pythagorean triangles with sides M, P, Q (so $M^2 = P^2 + Q^2$) such that $P+Q$ is a square and $M - (P-Q)$ is twice a square. *J. W. S. Cassels.*

Ginatempo, Nicola. Problemi di analisi indeterminata in n variabili. Atti. Soc. Peloritana Sci. Fis. Mat. Nat. 1 (1955), 15–25.

"La risoluzione per via geometrica dei problemi di analisi indeterminata è feconda di risultati estremamente facilit a ritrovarsi". This dictum is illustrated by finding the general solution in integers of (I) $x_1^2 + \cdots + x_n^2 = x_0^2$, (II) the simultaneous pair

$$x_1^2 + \cdots + x_n^2 = x_0^2, \quad x_1 + \cdots + x_n = x_0,$$

and (III) $x_0/x_1 + \cdots + x_n/x_0 = 1/h$, where h is a given integer. The proofs are routine except that that of (II) is egregiously clumsy, the obvious way being to set $x_1 = x_0 + tm_1$, $x_j = tm_j$ ($1 < j \leqq n$), where the m_j are parameters subject to $\sum m_j = 0$ and $t = -2m_1 x_0 / \sum m_j^2$.
J. W. S. Cassels (Cambridge, England).

Ankeny, N. C. Sums of three squares. Proc. Amer. Math. Soc. 8 (1957), 316–319.

The author gives a proof of Legendre's theorem that every positive integer m which is not of the form $4^a(8n+7)$ is representable as a sum of three integral squares; this proof uses only Minkowski's theorem on the existence of an integer point in a convex body, Dirichlet's theorem on the existence of primes in an arithmetic progression, and simple facts on quadratic reciprocity and on representation by two squares. The details are given for $m \equiv 3 \pmod 8$ and square-free, the modifications for other cases being briefly indicated. The essential idea is to represent m as

$$R^2 + 2(qx^2 + bxy + hy^2),$$

where R is a linear form in x, y, z, and q, b, h are suitable integers satisfying $b^2 - 4qh = -m$. The construction of these is such as to permit a proof that $qx^2 + bxy + hy^2$ has no prime factor $\equiv 3 \pmod 4$ dividing it to an odd power exactly, whence it follows that $2(qx^2 + bxy + hy^2)$ is a sum of two squares. *H. Davenport* (London).

Srivastava, Om Prakash. On the number of representations as sum of four squares of numbers of the form $4^a(8b+7)$. J. Sci. Res. Banaras Hindu Univ. 6 (1955–56), 278–285.

The author investigates the numbers which have precisely three partitions into four positive squares. These numbers are 55, 79, 95 and $4^k m$, where $k > 0$ and $m = 7$, 15 and 23. These facts are applied in an interesting way to the Tarry-Escott problem to derive three sets of eight numbers having equal sums of kth powers for $k = 1, 2, 3$. The same questions are answered about numbers which have precisely h partitions into four positive squares for $h = 1$, 2 and 4. [See also Lehmer, Amer. Math. Monthly 55 (1948), 476–481; MR 10, 182.] The reader may be assisted in following the paper if it is here noted that "$s(n)$" denotes the sum of the odd divisors of n and that "representation" usually means "partition".
D. H. Lehmer (Berkeley, Calif.).

Sastry, S. Representation of numbers in the form $n = \pm x_1^{k_1} \pm x_2^{k_2} \pm \cdots \pm x_s^{k_s}$. J. Sci. Res. Banaras Hindu Univ. 6 (1955–56), 214–216.

By purely algebraic methods the author proves that every positive integer is of the form

$$x^4 - y^4 - z^3 + t^3 + \varepsilon^3 \quad (\varepsilon = 0, 1),$$

and also of the form

$$x^5 + y^5 - 2z^5 + w^5 - r^3 - s^3 + t^3.$$

These results continue those of the author's previous paper [same J. 6 (1955–56), 87–89; MR 17, 827].
D. H. Lehmer (Berkeley, Calif.).

Sierpiński, W. What we know and what we do not know about decomposition of natural numbers into a sum of squares, cubes, and fourth powers. Prace Mat. 2 (1956), 56–64. (Polish)

Der Artikel behandelt im Wesentlichen verschiedene Fragen über die Existenz und die Anzahl von Zerlegungen der natürlichen Zahlen n in Summen von der Gestalt $n = x_1^r \pm x_2^r \pm \cdots \pm x_s^r$, wo r, s einige kleine natürliche Zahlen sind und wo $x_1, \cdots, x_s$ entweder ganze oder natürliche oder verschiedene ganze nichtnegative oder verschiedene natürliche Zahlen sind. Der Verfasser referiert über die älteren, neueren und die neuesten Resultate in diesem Gebiete und gibt einige bis jetzt noch nicht gelöste Probleme aus diesem Ideenkreise an.
V. Knichal (Prag).

Noguera Barreneche, Rodrigo. Quadratic forms in identities. Stvdia. Rev. Univ. Atlantico 2 (1957), nos. 11–12, 127–132. (Spanish)

Various new theorems of the following type: if p is a prime of the form $sk+t$, then all integral solutions of $cx^2+dy^2=p$ are found from identities of the form $f(a, b)=cx^2+dy^2$, where f is a quadratic polynomial in a and b, with coefficients depending on the given constants s, t, c, d.

Yin, Wen-Lin. Note of the representation of large integers as sums of primes. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 793–795 (1957).

The author proves by elementary methods that every sufficiently large integer is the sum of at most 18 primes. This improves a previous result of Shapiro and Warga [Comm. Pure Appl. Math. 3 (1950), 153–176; MR 12, 244] who proved that every sufficiently large integer is the sum of at most 20 primes. *P. Erdős* (Haifa).

Dempster, A. P. The minimum of a definite ternary quadratic form. Canad. J. Math. 9 (1957), 232–234.

The author gives a very simple determination of the lattice in Euclidean space of three dimensions of smallest cell for given minimum distance between lattice points, using extremely elementary geometrical methods.

J. A. Todd (Cambridge, England).

Zeckendorf, E. L'équation de Fermat. Bull. Soc. Roy. Sci. Liège 25 (1956), 414–425.

Noguera Barreneche, Rodrigo. General solution of the algebraico-exponential equation $X^v+Y^v=Z^v$. Stvdia. Rev. Univ. Atlantico 2 (1957), nos. 11–12, 119–126. (Spanish)

Vinogradov, A. I. On numbers with small prime divisors. Dokl. Akad. Nauk SSSR (N.S.) 109 (1956), 683–686. (Russian)

Let $F(x, z)$ denote the number of positive integers, not exceeding x, whose maximum prime factor does not exceed z. Many authors have estimated $F(x, z)$ in recent years, notably de Bruijn [Nederl. Akad. Wetensch. Proc. 53 (1950), 803–812; MR 12, 11]. Writing $z=x^\alpha$ where (λ is a certain positive absolute constant)

$$\frac{\lambda(\log\log x)^{7/4}}{(\log x)^{1/4}}\leq\alpha\leq\frac{1}{e},$$

the author obtains an asymptotic formula for $F(x, z)$, which is sharper than previous estimates. The formula is too long to be quoted here. *S. Chowla.*

Pellegrino, Franco. Lineamenti di una teoria delle funzioni aritmetiche. I. Rend. Mat. e Appl. (5) 15 (1956), 469–504 (1957).

This is the first of a series of papers giving proofs for the article reviewed in MR 15, 779.

See also: Lehmer, p. 7; Eichler, p. 17; Davenport, p. 18; Blaney, p. 19.

Analytic Theory of Numbers

Ricci, Giovanni. Sull'insieme dei valori di condensazione del rapporto $(p_{n+1}-p_n)/\ln p_n$ $(n=1, 2, 3, \cdots)$. Riv. Mat. Univ. Parma 6 (1955), 353–361.

This is essentially an Italian version of pp. 93–106 of Colloque sur la Théorie des Nombres, Bruxelles, 1955 [Thone, Liège, 1956; MR 18, 112].

P. Erdős (Haifa).

Nanda, V. S. Bipartite partitions. Proc. Cambridge Philos. Soc. 53 (1957), 273–277.

Denote by $p(m, n)$ the number of partitions of the bipartite number (m, n). Auluck [same Proc. 49 (1953), 72–83; MR 14, 726] gave asymptotic formulas for $p(m, n)$ in the cases when m is fixed and $n\rightarrow\infty$ and when m and n are of the same order. In this paper the author gives a formula for $m=o(n^{1/4})$. In this case

$$3^{\frac{1}{2}}4n(m!)p(m, n)\sim(6n\pi^{-2})^{m/2}\exp\{\pi(2n/3)^{\frac{1}{2}}\}.$$

This result follows by splitting the generating function of $p(m, n)$ into various sub-generating functions, of which only one makes an essential contribution when $m=o(n^{\frac{1}{4}})$. Use is then made of the Hardy-Ramanujan asymptotic formula for $p(n)$ with an m-fold summation process. One's hope of using the above formula in a special case is not heightened by observing that it gives

$$p(2, 50)\sim3{\cdot}3070{\cdot}10^6$$

whereas $p(2, 50)=5569166$. *D. H. Lehmer.*

Bennett, J. H. Partitions in more than one dimension. J. Roy. Statist. Soc. Ser. B. 18 (1956), 104–112.

Several important properties are derived for the new partitional functions introduced by Fisher [Proc. Roy. Soc. London. B. 136 (1950), 509–520; MR 11, 710] in his enumeration of the number of partitions of an integer in an arbitrary number of dimensions. These functions readily lead to the enumeration of the number of partitions of an integer in more than one dimension and with given marginal partitions. This enumeration is given for bipartitions of the integers 1 to 8 inclusive. (Author's summary.) *N. J. Fine* (Philadelphia, Pa.).

Wright, E. M. The number of partitions of a large bipartite number. Proc. London Math. Soc. (3) 7 (1957), 150–160.

A bi-partite number (m, n) is a two-dimensional vector whose components m, n are non-negative rational integers. A partition of (m, n) is a solution of the vector equation

$$\sum_j (m_j, n_j)= (m,n)$$

in bi-partite numbers other than (0, 0), the order being irrelevant. Let $p_1(m, n)$ be the numbers of partitions of (m, n) and $p_2(m, n)$ the number of such partitions in which no part has a zero component. Also let $p_3(m, n)$ be the number of partitions of (m, n) into different parts and $p_4(m, n)$ the number of such partitions without zero components. On the assumption that $A_1<m/n<A_2$, where A_1 and A_2 are positive constants, an asymptotic formula for $p_s(m, n)$ is obtained ($s\leq4$) which is too complicated to quote here. This formula involves rational functions and logarithms of m and n and a definite integral which does not appear to be expressible, in general, in terms of elementary functions. For the case $s=1$ an equivalent result was obtained by F. C. Auluck [Proc. Cambridge Philos. Soc. 49 (1953), 72–83; MR 14, 726].

R. A. Rankin (Glasgow).

★**Cheema, Mohinder Singh. Tables of partitions of Gaussian integers, giving the number of partitions of $n+im$. (Under the guidance of Hansraj Gupta.)** Nat. Inst. Sci. India: Mathematical tables, Vol I. National Institute of Sciences of India, New Delhi, 1956. xii+67 pp.

This work contains a main table of the number $B(n, m)$ of partitions of the bipartite number (n, m) into unrestricted non-negative summands (r, s). That is, $B(n, m)$ is the number of ways in which the Gaussian integer $n+im$ can be written as a sum of non-zero Gaussian integers $r+is$ in the first quadrant, the order of the summands being immaterial. Thus

$$\sum B(n, m)x^n y^m = \prod(1-x^r y^s)^{-1}.$$

In particular $B(n, 0)=p(n)$, the number of unrestricted partitions of n. The table gives $B(n, m)$ for $n\leq 50$, $m\leq 50$. The rest of the volume is devoted to the auxiliary function $B_k(n, m)$ which may be defined for $m\geq k$ as the number of partitions of $n+im$ into real integers and Gaussian integers with imaginary parts $\geq k$. ($B_k(n, m)=0$ when $0<m<k$). The table of $B(n, m)=B_1(n, m)$ was constructed from $B_k(n, m)$ starting with $k=50$ and working k down recursively. The tabulated values of $B_k(n, m)$ are given as follows

$$2\leq k\leq 15,\ 3k\leq m\leq 50$$
$$k=16,\ 32\leq m\leq 50$$
$$k=17,\ m=50$$

There is also a table of $p(n)$ and $\sum_{k=0}^{n} p(k)=B(n, 1)$ for $n\leq 50$ on page 65. *D. H. Lehmer* (Berkeley, Calif.).

★**Delange, Hubert. Sur la distribution des valeurs de certaines fonctions arithmétiques.** Colloque sur la Théorie des Nombres, Bruxelles, 1955, pp. 147–161. Georges Thone, Liège; Masson and Cie, Paris, 1956.

This lecture deals with distribution properties of the functions $\omega_E(n)$ and $\Omega_E(n)$, defined by a set E of prime numbers in the following way: $\omega_E(n)=\sum_{p\in E, p|n} 1$, $\Omega_E(n)=\sum_{p\in E, p|n} \lambda(n)$ ($\lambda(n)$ is the maximal λ such that $p^\lambda|n$). It is assumed that the set E has the property (H): There is a constant α ($0<\alpha\leq 1$) such that $\sum_{p\in E} p^{-s}+\alpha\log(s-1)$ is regular throughout the closed halfplane $\mathrm{Re}\, s\geq 1$. For example, the set of all p has the property (H), and so has the set of all p in any given arithmetic progression. Using some of his work on Tauberian theorems [Ann. Sci. Ecole Norm. Sup. (3) **71** (1954), 213–242; MR **16**, 921], the author shows that the values of $\omega_E(n)$ are equally distributed over the residue classes mod q, for integer q. The same thing holds for $\Omega_E(n)$. Moreover, if f and g are functions ω (or Ω) belonging to disjoint sets E_1 and E_2, both having property (H), then the residue classes of $f(n)$ mod q and $g(n)$ mod q are statistically independent. And, if only those n are considered for which $f(n)$ equals a given integer h, then $g(n)$ is still equally distributed over the residue classes mod q. In some cases these results remain true if n is assumed to run over all squarefree integers >0 instead of over all integers >0. In the proofs, a central role is played by sums $\sum_1^\infty z^{f(n)}n^{-s}$, where z is a root of unity. [For details, the author refers to Ann. Sci. Ecole Norm. Sup. (3) **73** (1956), 15–74; MR **18**, 720.] It is conjectured that the condition (H) can be replaced by the more natural condition $\sum_{p\in E} p^{-1}=\infty$. *N. G. de Bruijn.*

Addison, A. W. A note on the compositeness of numbers. Proc. Amer. Math. Soc. **8** (1957), 151–154.

The "compositeness" of the number $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$ is defined by $\Omega(n)=\alpha_1+\alpha_2+\cdots+\alpha_m$. If for any $q\geq 3$ we partition the integers into q classes $\{C_{q,i}\}$ ($i=0, 1, \cdots, q-1$) according to whether $\Omega(n)\equiv 0, 1, \cdots, q-1 \pmod q$, and let $C_{q,i}(x)$ be the corresponding counting functions, the author proves that

$$C_{q,i}(x)-x/q=\Omega_\pm(x/\log^r x)\ (i=0, 1, \cdots, q-1),$$

where $r=1-\cos(2\pi/q)$. The leading term x/q, with error of $o(x)$, has already been established by S. Selberg [Math. Z. **44** (1938), 306–318] and S. S. Pillai [Proc. Indian Acad. Sci. Sect. A. **11** (1940), 13–20; MR **1**, 293]. *S. Ikehara* (Tokyo).

Steinhaus, H. On golden and iron numbers. Zastos. Mat. **3** (1956), 51–65. (Polish. Russian and English summaries)

Durch die Beziehungen $z_n=nz$, $s_n=n(1-z)$, wo $z=\frac{1}{2}(5^{\frac{1}{2}}-1)$ ist, definiert zuerst der Verf. die Folge $\{z_n\}$ von „goldenen" und die Folge $\{s_n\}$ von „silbernen" Zahlen, und benutzend die Eigenschaft, dass diese Zahlen sehr gleichmässig mod 1 verteilt sind und dass deshalb auch $n^{-1}\sum_{k=1}^{n} f(z_k)$ für beliebige integrierbare Funktion $f(x)$ mit der Periode 1 sehr gut gegen $\int_0^1 f(x)dx$ konvergiert, zeigt er den Vorteil ihrer Benutzung zum statistischen Zwecke. Für eine gegebene natürliche Zahl N definiert er weiter die endliche Folge von „eisernen" Zahlen $\{c_n\}$ ($n=1, \cdots, N$) als eine gewisse Permutation von Zahlen $1, \cdots, N$ so, dass die Folge von Zahlen $c_nz-[c_nz]$ wachsend sei ($[\mu]$ bedeutet dabei die grösste ganze Zahl $\leq\mu$). Es wird der Vorteil dieser Folge bei ihrer Benutzung zu den Zwecken der Stichprobe und zur Qualitätskontrolle gezeigt. Es handelt sich nämlich um eine in gewissen Sinne universale Folge, die repräsentative Auswahlen (samples) aus beliebiger Anzahl $k\leq N$ von Elementen gibt. Der Artikel ist durch den Beweis des Satzes beendet, dass für jede irrationale Zahl z und für jede natürliche Zahl n die Punkte $z, 2z, \cdots, nz$ die Zahlenachse mod 1 in Intervalle zerteilen, unter denen höchstens drei verschiedene Länge haben können. *V. Knichal* (Prag).

See also: Srivastava, p. 15; Sastry, p. 15; Vinogradov, p. 16; Carlitz, p. 29.

Theory of Algebraic Numbers

Eichler, M. Berichtigung zu der Arbeit "Zur Zahlentheorie der Quaternionen-Algebren". J. Reine Angew. Math. **197** (1957), 220.

A slight reformulation of the proof of Theorem 10 in the article with the given title reviewed in MR **18**, 297.

Newman, Morris. An inclusion theorem for modular groups. Proc. Amer. Math. Soc. **8** (1957), 125–127.

From a generalization [I. Reiner and J. D. Swift, Pacific Math. **6** (1956), 529–540; MR **18**, 565] of his earlier theorem [Duke Math. J. **22** (1955), 25–32; MR **16**, 801], the author proceeds to consider G_R, the group of matrices for which the elements $\alpha, \beta, \gamma, \delta$ are members of a ring R in an algebraic number field and $\alpha\delta-\beta\gamma=1$. Then he lets $G_R(\mathfrak{m}, \mathfrak{n})$ be the group of matrices for which $\beta\equiv 0 \bmod \mathfrak{m}$, $\gamma\equiv 0 \bmod \mathfrak{n}$, for ideals satisfying $(\mathfrak{m}, 6)=(\mathfrak{n}, 6)=(\mathfrak{m}, \mathfrak{n})=1$. Then if H is a sub-group of G_R containing $G_R(\mathfrak{m}, \mathfrak{n})$, he shows $H=G_R(\mathfrak{m}_1, \mathfrak{n}_1)$, where $\mathfrak{m}_1$ and $\mathfrak{n}_1$ divide $\mathfrak{m}$ and $\mathfrak{n}$ respectively. The restriction involving "6" can be removed for the rational field, but the author shows that relative primeness can not be removed. *H. Cohn* (St. Louis, Mo.).

Eichler, Martin. Modular correspondences and their representations. J. Indian Math. Soc. (N.S.) 20 (1956), 163-206.

The author observes that modular correspondences [cf. Hecke, Math. Ann. **114** (1937), 1–28] are basically connections between certain subgroups of the modular group. He suggests the possibility of generalization in replacing the modular group by the other appropriate groups, e.g., the groups of units of orders of central simple algebras over algebraic number fields. He here develops this possibility for the groups of units of certain orders of an indefinite quaternion algebra Q over the rational field, using many results of his earlier papers [e.g., J. Math. **195** (1955), 127–151; MR **18**, 297].

Let T be an order of Q of "square-free level," and let ε be a unit of T of norm 1. Then ε can be represented as a 2 by 2 matrix with elements in a real splitting field K of Q. This matrix defines a transformation, $\tau\rightarrow\varepsilon\circ\tau$, of the complex upper half plane into itself in the usual way. The transformations belonging to the group U of units of T of norm 1 constitute a faithful representation of $U/\{1, -1\}$. The transformation group is properly discontinuous in the upper half plane and by the identification of equivalent points defines a closed surface S_T with a hyperbolic metric (with appropriate attention to elliptic and parabolic vertices, if any). The author computes the genus of S_T and the number of elliptic vertices (of orders 2 and 3). "Modular correspondences" arise by considering two orders T and T' of rank 4 of Q, and $T^*=T\cap T'$. Let the respective groups of units of norm 1 be U, U' and U^*, transformation groups $\Gamma_T, \Gamma_T', \Gamma_{T^*}$, surfaces S_T, S_T', S_{T^*}. Then $[\Gamma_T:\Gamma_{T^*}]$ and $[\Gamma_T':\Gamma_{T^*}]$ are finite, say d' and d, so that $\Gamma_T=\sum_{i=1}^{d'}\Gamma_{T^*}\varepsilon_i', \Gamma_T'=\sum_{i=1}^{d}\Gamma_{T^*}\varepsilon_i$. By the definition that the point $\Gamma_T'\circ\tau$ of $S_{T'}$ is covered by the points $\Gamma_{T^*}\circ(\varepsilon_i\circ\tau)=\Gamma_{T^*}\circ\tau_i$ $(i=1, \cdots, d)$ of S_{T^*}, the latter appears as a d-sheeted covering surface of S_T'. Conversely, $\Gamma_T'\circ\tau$ is the trace in S_T' of the points $\Gamma_{T^*}\circ\tau_i$ in S_{T^*}. Finally, the set-theoretical union of the traces P_i in S_T of all points P_i^* of S_{T^*} covering a given point P' in S_T', is called the geometrical correspondence of S_T' to S_T. Applying this with $T'=\nu T\nu^{-1}$, $\nu\in Q$, ν of positive norm, one has $\Gamma_T'=\nu\Gamma_T\nu^{-1}$. The 2 by 2 matrix over K which represents ν maps S_T onto S_T' and the geometrical correspondence of S_T' to S_T leads to a correspondence of S_T to itself, belonging to the element ν. The set of such correspondences generates, under appropriate combining definition, an associative ring with unit element which the author shows has representations as endomorphisms of Betti groups. In this connection he gives a proof of a fixed-point theorem of Lefschetz [Algebraic topology, Amer. Math. Soc. Colloq. Publ., v. 27, New York, 1942; MR 4, 84]. *R. Hull* (Los Angeles, Calif.).

Nakano, Noboru. Über den Primäridealquotienten im Unendlichen algebraischen Zahlkörper. J. Sci. Hiroshima Univ. Ser. A. 18 (1955), 257–269.

Let $\mathfrak{q}$, $\mathfrak{q}'$ be primary ideals in the principal order of an infinite algebraic number field. The author considers the structure of the quotient $\mathfrak{q}:\mathfrak{q}'$ for $\mathfrak{q}\subset\mathfrak{q}'$ and gives a necessary and sufficient condition for the product decomposition $\mathfrak{q}=\mathfrak{q}'\mathfrak{q}''$ using the classification (i)–(iv) of primary ideals in a former paper [same J. 17 (1954), 321-343; MR 16, 907]. *Y. Kawada* (Tokyo).

Nakano, Noboru. Idealtheorie im Stiemkeschen Körper. J. Sci. Hiroshima Univ. Ser. A. 18 (1955), 271–287.

An infinite algebraic number field is called a Stiemke-field if every ideal of the principal order is divisible by only finitely many prime ideals [E. Stiemke, Math. Z. **25** (1926), 9–39]. The principal order of a Stiemke field provides a simple example in which the ascending chain condition for ideals does not hold. The author develops the ideal theory in such a field without the help of valuation theory. *Y. Kawada* (Tokyo).

Nakano, Noboru. Über die Multiplikativeigenschaft der Ideale in unendlichen algebraischen Zahlkörpern. J. Sci. Hiroshima Univ. Ser. A. 19 (1956), 439–455.

Let $\mathfrak{K}$ be an infinite algebraic number field and $\mathfrak{K}=\bigcup_{n=1}^{\infty}\mathfrak{K}_n$, $\mathfrak{K}_1\subset\mathfrak{K}_2\subset\cdots$, with $\mathfrak{K}_n$ $(n=1, 2, \cdots)$ finite. Let $\mathfrak{o}$ and $\mathfrak{o}_n$ be the principal orders of $\mathfrak{K}$ and $\mathfrak{K}_n$ respectively. For "everywhere finite" ideals $\mathfrak{a}$, $\mathfrak{b}$ $(\mathfrak{a}\subset\mathfrak{b})$ of $\mathfrak{o}$ Krull [Math. Z. 31 (1930), 527–557] gave a necessary and sufficient condition that the product decomposition $\mathfrak{a}=\mathfrak{b}\mathfrak{c}$ holds in terms of valuation theory. The author proves here that $\cdots\subset\mathfrak{a}_\nu:\mathfrak{b}_\nu\subset\mathfrak{a}_{\nu+1}:\mathfrak{b}_{\nu+1}\subset\cdots$ for $\nu\geqq N$ with sufficiently large N (where $\mathfrak{a}_\nu=\mathfrak{a}\cap\mathfrak{o}_\nu$, $\mathfrak{b}_\nu=\mathfrak{b}\cap\mathfrak{o}_\nu$) is also necessary and sufficient under the same assumption. For ideals $\mathfrak{a}$, $\mathfrak{b}$ which are not "everywhere finite" he assumes the following condition: let $\mathfrak{q}$ be the isolated primary component of $\mathfrak{a}$ belonging to a prime ideal $\mathfrak{p}$, and N be a sufficiently large number independent of $\mathfrak{p}$; then for any $\nu\geqq N$, $\mathfrak{q}\cap\mathfrak{o}_\nu$ is the isolated primary component belonging to $\mathfrak{p}\cap\mathfrak{o}_\nu$. Then for such ideals $\mathfrak{a}$, $\mathfrak{b}$ of $\mathfrak{o}$ he gives a necessary and sufficient condition for the product decomposition $\mathfrak{a}=\mathfrak{b}\mathfrak{c}$ in a similar form as above. *Y. Kawada* (Tokyo).

Butts, H. S.; and Mann, H. B. Corresponding residue systems in algebraic number fields. Pacific J. Math. **6** (1956), 211–224.

Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be number fields and $\mathfrak{a}_1$ and $\mathfrak{a}_2$ ideals in $\mathfrak{F}_1$ and $\mathfrak{F}_2$ respectively. We say $\mathfrak{a}_1=\mathfrak{a}_2$ if they generate the same ideal $\mathfrak{a}$ in $\mathfrak{F}_1\cup\mathfrak{F}_2$, and $\mathfrak{F}_1$ and $\mathfrak{F}_2$ have corresponding residue systems modulo $\mathfrak{a}$ if for every integer α_1 of $\mathfrak{F}_1$ there exists an integer α_2 of $\mathfrak{F}_2$ such that $\alpha_1\equiv\alpha_2$ (mod $\mathfrak{a}$) and conversely. The authors prove several necessary or sufficient conditions for this problem, and the case of Kummer extensions $\mathfrak{F}_1=\mathfrak{F}(\mu_1^{1/q})$ and $\mathfrak{F}_2=\mathfrak{F}(\mu_2^{1/q})$ and of a power of a prime ideal is considered in detail. *Y. Kawada* (Tokyo).

Ankeny, N. C.; and Chowla, S. On the divisibility of the class number of quadratic fields. Pacific J. Math. **5** (1955), 321–324.

It is shown that the number N of square-free integers of the form $d=3^g-x^2$ with $2|x$, $0<x<(2\cdot3^{g-1})^{1/2}$ for g sufficiently large and even, is $\geqq\frac{1}{25}3^{g/2}$. Then it is proved that the class number h of the imaginary quadratic field $R(\sqrt{-d})$ is a multiple of g. It follows that there are infinitely many quadratic imaginary fields each with a class number divisible by g. It is further shown that the real quadratic field $R(\sqrt{d})$ with $d=n^{2g}+1$ $(n>4)$ is divisible by g if d is square free. *C. Arf* (Istanbul).

Geometry of Numbers

★ **Davenport, H. Simultaneous Diophantine approximation.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 9–12. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

A brief survey of the little that is known about the approximation to a set of irrationals $\theta_1, \cdots, \theta_n$ by rational

fractions $p_1/q, \cdots, p_n/q$ with a common denominator, other than the so-called "metrical theory".
J. W. S. Cassels (Cambridge, Mass.).

Davenport, H. Indefinite quadratic forms in many variables. Mathematika 3 (1956), 81–101.

This paper is devoted to a proof of the following theorem. Let $Q(x_1, \cdots, x_n)$ be an indefinite quadratic form with real coefficients and index r. Suppose that

$$(3)\qquad r \geqq 37 \text{ and } n-r \geqq 37.$$

Then for each $\varepsilon>0$, there exist integers $x_1, \cdots, x_n$, not all 0, such that

$$(2)\qquad |Q(x_1, \cdots, x_n)|<\varepsilon.$$

Using results of Oppenheim [Quart. J. Math. Oxford Ser. (2) 4 (1953), 54–59, 60–66; MR 14, 955] it immediately follows that a quadratic form satisfying (3) is either a multiple of a form with integral coefficients or it assumes values that are everywhere dense among the real numbers. The proof uses: 1) a modification of the Hardy-Littlewood method used by Davenport and Heilbronn [J. London Math. Soc. 21 (1946), 185–193; MR 8, 565] in proving (2) for $n=5$ where Q lacks cross product terms, and 2) a recent estimate of Cassels [Proc. Cambridge Philos. Soc. 51 (1955), 262–264; 52 (1956), 602; MR 16, 1002; 17, 380].
B. W. Jones (Boulder, Colo.).

Blaney, Hugh. On the Davenport-Heilbronn theorem. Monatsh. Math. 61 (1957), 1–36.

Let $a, b, c, d, \lambda_0, \mu_0$ be real numbers, with $ad-bc=\pm 1$, and let

$$Q(x, y)=(ax+by+\lambda_0)(cx+dy+\mu_0).$$

It follows from a result of Davenport and Heilbronn [J. London Math. Soc. 22 (1947), 53–61; MR 9, 413] that there exist integers x, y such that $0<Q\leqq 1$. The author has already proved [Thesis, Univ. of London 1949] the existence of a constant ξ $(0<\xi<1)$ such that the inequality $\xi \leqq Q \leqq 1$ is soluble in integers x, y, and he now shows that ξ can be chosen as:

$$\xi=\tfrac{1}{8}(-11+\sqrt{126})=[0, \overline{8, 1, 8, 11}]=0.11248\cdots.$$

With this value, the strict inequality $\xi<Q<1$ is soluble unless Q is equivalent (in an appropriate sense) to one of the three forms

$$xy,\ \tfrac{1}{8}y(2x+y+1),\ \tfrac{1}{2}y(2x+(1+2\xi)y+1).$$

The above value for ξ is best possible in the stronger sense that for any small $\delta>0$ there exist infinitely many forms Q such that the inequality $\xi+\delta<Q<1$ is insoluble. The proof of the results just stated is difficult, and requires the consideration of many cases. The treatment is on different lines according as a/b and c/d are both irrational, or not.
H. Davenport (London).

Ehrhart, Eugène. Sur une famille de polyèdres. C. R. Acad. Sci. Paris 244 (1957), 434–437.

Ehrhart, Eugène. Sur les polyèdres homothétiques. C. R. Acad. Sci. Paris 244 (1957), 157–160.

These articles are continuations of previous notes [same C.R. 242 (1956), 1570–1573, 1844–1846, 2217–2219; 243 (1956), 347–349; MR 17, 948; 18, 383].
J. H. H. Chalk (London).

See also: Dempster, p. 16.

ANALYSIS

★ **Garcia, Godofredo; y Rosenblatt, Alfred. Análisis algebraico. [Algebraic analysis.]** Sanmarti y Compania, Lima, 1955. 252 pp.

This is an introduction to the real number system. There are many quotations from Greek mathematicians (the Greek titles being filled in by hand, with occasional slips, e.g. on p. 4), and also from the moderns: Dedekind, Cantor, Weierstrass, Peano, Russell, and many others. The philological and historical setting makes a very pleasant impression. The chapter headings are: real numbers, natural numbers, elements of sets of points on a straight line, sequences and series, powers and logarithms. Though written in an unhurried way, the book contains a great deal of mathematical information. *S. H. Gould.*

★ **Lichnerowicz, André. Lineare Algebra und lineare Analysis.** Hochschulbücher für Mathematik, Band 28. VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. xi+303 pp.

A translation by Gisela Päch and Almut Welz from the French of the work reviewed in MR 9, 414.

★ **Толстов, Г.П. [Tolstov, G. P.] Курс математического анализа. Том I. [Course of mathematical analysis. Vol. I.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 551 pp. 11.40 rubles.

The first volume has ten chapters. Some of the headings are: real numbers, theory of limits; derivative and differential; elements of differential geometry on a surface; complex numbers and complex functions; indefinite integral; definite integral; integration of discontinuous functions, improper integrals.

Functions of Real Variables

Iosifescu, Marius. Sur un théorème de A. Marchaud. Com. Acad. R. P. Romîne 6 (1956), 1169–1171. (Romanian. Russian and French summaries)

The theorem of Marchaud [Fund. Math. 20 (1933), 105–116] is: "toute fonction continue qui prend chacune de ses valeurs un nombre fini de fois possède une dérivée presque partout; la fonction peut être modifiée sur un ensemble de mesure aussi petite qu'on le veut, de manière à obtenir une fonction continue à variation bornée".

The present author gives a direct proof of this theorem, which had been proved indirectly by Marchaud. He also proves that "le théorème est vrai pour toute fonction, continue ou non, ayant l'ensemble des points de non monotonie de mesure nulle".

Marcus, S. Sur la détermination d'une fonction partiellement continue par les valeurs prises sur un ensemble dense. II. Com. Acad. R. P. Romîne 6 (1956), 985–987. (Romanian. Russian and French summaries)

Three theorems, of which the first is: if a function $f(x_1, x_2, \cdots, x_n)$ of n variables, defined on a domain G, vanishes on a set E which is everywhere dense in G and if f is continuous in each of its variables separately, then f vanishes identically in G. Compare part I, which was reviewed in MR 17, 1064.

Bečvář, Jiří; and Nekvinda, Miloslav. Extremals of functions of two and several variables. Časopis Pěst. Mat. 81 (1956), 267–271. (Czech)

The standard condition for $F(x, y)$ to have an extremum

at a stationary point (a, b) is that $D=F_{xx}F_{yy}-F_{xy}^2>0$ at (a, b). The authors observe that it is sufficient that $D>0$ near (a, b), when $D=0$ at (a, b). If $D<0$ near (a, b), then F does not have an extremum there. The extension to n variables imposes concavity or convexity near the stationary point. *F. V. Atkinson.*

Landau, B. V. Use of a "D-backwards" operator. Math. Gaz. 41 (1957), 127–129.

An operator $\mathcal{D}$ is introduced to indicate differentiation of the function preceding it and is used for convenient evaluation of expressions like $(D^2+a^2)^{-1} \cos ax$.

de Rham, G. Sur un exemple de fonction continue sans dérivée. Enseignement Math. (2) 3 (1957), 71–72.

The solutions of the functional equation
$$F(x)-bF(ax)=g(x),$$
for various choices of the function $g(x)$ and the constants a and b, provide examples of continuous, non-differentiable functions.

Ascoli, Guido. Sopra una larga estensione di una classica proprieta delle funzioni armoniche. Rev. Un. Mat. Argentina 17 (1955), 3–8 (1956).

Sia I un insieme di elementi x, K un sottinsieme di I, (L) una classe lineare di funzioni reali $f(x)$ definite in I. Si supponga che per ogni $f \in (L)$ sia $\sup_{x\in I} f=\sup_{x\in K} f$ e quindi anche, per la linearità di (L), $\inf_{x\in I} f=\inf_{x\in K} f$. In tali ipotesi vale il teorema seguente: se $f_1(x)$, $f_2(x)$, $\cdots$, $f_n(x) \in (L)$ e se per $x \in K$ il punto $Q\equiv[f_1(x), f_2(x), \cdots, f_n(x)]$ appartiene ad un corpo convesso C (dello spazio euclideo S_n), allora anche per $x \in I$ si ha $Q \in C$. Di questo teorema sono mostrate varie applicazioni, qualcuna riguardante le funzioni armoniche. *A. Ghizzetti* (Roma).

Babenko, K. I. On a new problem of quasi-analyticity and on the Fourier transform of entire functions. Trudy Moskov. Mat. Obšč. 5 (1956), 523–542. (Russian)

Continuing work of Šilov [Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 893–895; MR 17, 351] and Gelfand and Šilov [Uspehi Mat. Nauk (N.S.) 8 (1953), no. 6(58), 3–54; MR 15, 867] the author considers classes of functions on $(-\infty, \infty)$ which with all their derivatives are subject to bounds. The nature of the bounds on the derivatives of the Fourier transforms of these functions are considered. The results are extended to entire functions of several complex variables and an application is made to the following uniqueness theorem. Let $\partial u/\partial t=P(i^{-1}\partial/\partial x, t)u$, where u is a vector with m components and x one with n components. P denotes an $m\times m$ matrix with elements which are linear differential operators with respect to $x_1, \cdots, x_n$, and with coefficients which are continuous functions of t. Let p be the highest order of differentiation occurring in P. If $U(x, 0)=0$ and $|U(x, t)|<\exp(c|x|^q)$ where $q=p/(p-1)$, then $u(x, t)\equiv0$. *N. Levinson.*

See also: Stein, p. 23; Ford, p. 48.

Measure, Integration

Falevič, B. Ya. On a problem of N. N. Luzin. Moskov. Gos. Univ. Uč. Zap. 181. Mat. 8 (1956), 165–173. (Russian)

If M is a subset of the circumference of a circle and α_1 is a rotation of this set through an angle α_1, let M^{α_1} be the resulting image of this set, and let $M_1=M\cap M^{\alpha_1}$. If α_2 is similarly a rotation of M_1, put $M_2=M_1\cap M_1^{\alpha_2}$. This process could be repeated any number of times. Luzin once asked [Integral and trigonometric series, Gostehizdat, Moscow-Leningrad, 1951; MR 14, 2] whether, for any set M of the first category and measure zero, hereafter called a (1, 0)-set, there exists a finite sequence of rotations $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $M_{n-1}\cap M_{n-1}^{\alpha_n}$ is empty.

In this paper, Falevič constructs a (1, 0)-set on the circumference of a circle of radius $1/(2\pi)$ which gives a negative answer to the above question. This result naturally suggests a classification of (1, 0)-sets as follows (inductively): 1) The (1, 0)-set M is of order 1, if there exists a rotation α such that M_1 is empty. $1<n$) The (1, 0)-set M is of order n, if it is not of order $n-1$, but there exists a sequence of n rotations $\alpha_1, \cdots, \alpha_n$ such that M_n is empty. ∞) The (1, 0)-set M is of order ∞, if it is of no finite order, but there exists an infinite sequence of rotations $\alpha_1, \cdots, \alpha_n, \cdots$ such that $\bigcap_{k=0}^{\infty} M_k$ is empty $(M_0\equiv M)$. ω) The (1, 0)-set M is of order ω if the above $\bigcap_{k=0}^{\infty} M_k$ is not empty for any sequence of rotations. It is not clear whether sets of order ∞ really exist (Falevič surmises that they do); the set here constructed is of order ω.

In conclusion, the following theorem is demonstrated: For a (1, 0)-set to be of order ω, it is necessary and sufficient that, for any denumerable subset A (on the circle containing M), there exist a rotation x such that $M\cap M^x\supset A$. The (1, 0)-sets thus form a class of "universal" sets. The existence of sets of the second category and measure zero which are "universal" in the above sense follows from certain results of Borel (in his works on probability theory). *B. Dushnik.*

McMinn, Trevor J. Linear measures of some sets of the Cantor type. Proc. Cambridge Philos. Soc. 53 (1957), 312–317.

From the closed unit interval remove the central open interval of length $\frac{1}{3}$; from each of the two remaining closed intervals remove the central open interval of length $\frac{1}{3}(\frac{1}{3})^2$; from each of the four remaining closed intervals remove the central open interval of length $\frac{1}{3}(\frac{1}{3})^3$; from each of the eight remaining closed intervals remove the central open interval of length $\frac{1}{3}(\frac{1}{3})^4$; etc. Where $r>0$, let T_r be the planar set which is the Cartesian product of the resulting Cantor-type set with its magnification by factor r. The author determines the Carathéodory linear measure $L(T_r)$ of T_r for all positive r, and in particular shows that if $\frac{1}{2}\leqq r\leqq2$, then
$$L(T_r)=\operatorname{diam}(T_r)=(1+r^2)^{\frac{1}{2}}.$$
This sharpens and extends a result of Randolph's [J. London Math. Soc. 16 (1941), 38–42; MR 3, 226] to the effect that
$$3/\sqrt{5}\leqq L(T_1)\leqq\sqrt{2}.$$
Some other related results concerning Carathéodory linear measure and Gillespie linear measure are stated without proof. *T. A. Botts* (Charlottesville, Va.).

Mařík, Jan. A note on non-dense sets in E_m. Časopis Pěst. Mat. 81 (1956), 337–341. (Czech. Russian and English summaries)

Let E_m be Cartesian m-space, $B\subset E_m$, and
$$x=[x_1, \cdots, x_{m-1}] \in E_{m-1}.$$

Denote by $N(i, B)$ the set of all $x \in E_{m-1}$ for which the set

$$B \cap \{y : y = [x_1, \cdots, x_{i-1}, t, x_i, \cdots, x_{m-1}] \text{ for some } t \in E_1\}$$

is non-denumerable. Let $M(i, B)$ be the closure of $N(i, B)$, and let R^m be the class of closed $B \subset E_m$ for which each of $M(i, B)$, $1 \leq i \leq m$, has measure 0. These results are proved: (1) If $B \subset E_2$ is non-dense, if $N(1, B)$ has measure 0, and if $N(2, B)$ is of the first category, then B has measure 0. (2) If $B \in R^m$ is non-dense, then B has measure 0.

V. E. Beneš (Murray Hill, N.J.).

Glivenko, E. V. On measures of the Hausdorff type. Mat. Sb. N.S. **39(81)** (1956), 423–432. (Russian)

Let R be a metric space covered, in Vitali's sense, by a sequence of open sets $\{\Gamma_i\}$ such that diam $\Gamma_i \to 0$ as $i \to \infty$, and let $\alpha_i \geq 0$ $(i = 1, 2, \cdots)$. When M is any subset of R the author writes $\text{mes}_\varepsilon M$ for $\inf \sum_{n=1}^\infty \alpha_{i_n}$ with

$$M \subset \bigcup_{n=1}^\infty \Gamma_{i_n}$$

and diam $\Gamma_{i_n} < \varepsilon$, and he writes mes M for lim $\text{mes}_\varepsilon M$ as $\varepsilon \to +0$. He states the following theorem: Let an A-set M contained in R have a finite [infinite] measure. Then, for any $\varepsilon > 0$, there is a closed set $F \subset M$ such that

$$|\text{mes } F - \text{mes } M| < \varepsilon \ (\text{mes } F > 1/\varepsilon).$$

He deduces this from the following: If the set $M \subset R$ is the union of an increasing sequence of open sets $\{M_n\}$, then $\text{mes}_\varepsilon M_n \to \text{mes}_\varepsilon M$ as $n \to \infty$. However, this is not true in general. (Example: let R be the real line, let the Γ_i's be open intervals (r, r') with r and r' rational, let $\alpha_i = 2$ or 1 according as 0 is or is not in the closure of Γ_i, and let $M = (0, \frac{1}{2}\varepsilon)$, $M_n = (1/n, \frac{1}{2}\varepsilon)$; then $\text{mes}_\varepsilon M = 2$, however $\text{mes}_\varepsilon M_n = 1$.) Flaws in the author's argument will be found on pp. 426, 427. His theorem is similar to various known results [cf. M. E. Munroe, Introduction to measure and integration, Addison-Wesley, Cambridge, Mass., 1953, pp. 107-112; MR **14**, 734].

H. P. Mulholland.

Sion, Maurice. Variational measure. Trans. Amer. Math. Soc. **83** (1956), 205–221.

If f is a function on X to Y, ν is a measure on Y, and F a family of sets on X, the author defines a measure $\mu = V(F, f, \nu)$ on X, and establishes various properties of this measure. For example, any set of F is μ-measurable, and if ν is an outer measure and A is a ν-measurable set then ${}^*f(A)$ (the inverse image of A) is μ-measurable. He thus obtains conditions on the function f and class M of measures in order that f map a set, which is ν-measurable for all ν in M, into a set of the same kind.

U. S. Haslam-Jones (Oxford).

Zink, Robert E. A note concerning regular measures. Duke Math. J. **24** (1957), 127–135.

Where X is a topological space, and $(X, \mathbf{S}, \mu)$ is an associated measure space, and f is a non-negative measurable function on X, consider the measure ν on $\mathbf{S}$ defined for each E in $\mathbf{S}$ by

$$\nu(E) = \int_E f d\mu.$$

It is shown first that if μ is inner regular, then so is ν. The main theorem of the paper asserts that if μ is outer regular, then ν is outer regular if and only if, for each measurable set E on which f is integrable, there exists a measurable open set containing E on which f is integrable, and for every $\varepsilon > 0$ and $\delta > 0$ there is a measurable open set U containing the part of E not in the nucleus of f such that $\mu(\{x : f(x) > \delta\} \cap U) < \varepsilon$.

T. A. Botts.

Sucheston, Louis. A note on conservative transformations and the recurrence theorem. Amer. J. Math. **79** (1957), 444–447.

Let B be a σ-algebra of subsets of a set X, and let S be a subclass of B. Let T be a multi-valued transformation of X into itself. Denote by $T^{-1}A$ the set of points x such that $Tx \cap A \neq 0$, and assume that $T^{-1}A \in B$ whenever $A \in B$. T is called conservative if the relations $A \in B$ and $A \cap T^{-i}A = 0$ $(i \geq 1)$ imply $A \in S$. Simple proofs of the following theorems are given: (A) If S is closed under finite set unions and if T is conservative, then T^n is conservative for all $n \geq 1$. (B) If S is closed under countable set unions and if T is conservative, if $A \in B$ and if E denotes the set of points $x \in A$ such that $T^n x \cap A \neq 0$ for only finitely many positive integers n, then E belongs to S. If T is conservative in the above sense then it is conservative in the sense that $A \in S$ whenever $A \in B$ and the sets $T^{-i}A$ $(i \geq 0)$ are disjoint. It is remarked that for single valued transformations these two definitions are equivalent, but that for multi-valued transformations the recurrence theorem (B) is not in general true with the weaker definition.

J. C. Oxtoby (Bryn Mawr, Pa.).

★ **Pauc, Chr. Contributions à la théorie de la dérivation de fonctions d'ensemble.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 127–131. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Der erste Teil dieses Berichtes beschreibt neuere, durch C. A. Hayes und den Verf. [Canad. J. Math. **7** (1955), 221–274; MR **17**, 719] erzielte Ergebnisse in der auf de Possel zurückgehenden Theorie der punktweisen Differentiation sigma-additiver Mengenfunktionen in abstrakten Maßräumen mit Hilfe von Ableitungsbasen, d.h. Zuordnungen Moore-Smithscher Mengenfolgen zu Punkten des Raumes. Im zweiten Teil wird die vom Verf. [C. R. Acad. Sci. Paris **236** (1953), 1937–1939; MR **15**, 205] aufgestellte Theorie der Zellenfunktionen (verallgemeinerten Intervallfunktionen in abstrakten Maßräumen) von beschränkter Variation, in der an die Stelle der üblichen, punktweise gewonnenen Derivierten Radon-Nikodýmsche Integranden oder in $L^{(p)}$ gebildete Grenzfunktionen treten, skizziert. In dem hierbei zitierten Ergebnis des Ref. handelt es sich allerdings nicht um $L^{(1)}$-Derivierte, wie der Verf. sagt, sondern um wesentliche Derivierte. Schließlich werden die Grundlagen einer von D. Rutowitz und vom Verf. entwickelten verallgemeinerten Ward-Denjoyschen Theorie der Differentiation additiver Zellenfunktionen, die nicht notwendig von beschränkter Variation sind, angegeben. Die Gültigkeit dieser Differentiationssätze beruht auf einem von Rutowitz formulierten Gitteraxiom (axiome de grille) über die abstrakten Zellen, das eine geometrische Eigenschaft gewöhnlicher achsenparalleler Intervalle ausdrückt.

K. Krickeberg.

Steinhaus, H. Length, shape and area. Prace Mat. **2** (1956), 65–78. (Polish. Russian and English summaries)

A translation from the English of the paper reviewed in MR **16**, 121.

Dubins, L. E. Generalized random variables. Trans. Amer. Math. Soc. **84** (1957), 273–309.

Let $\mathscr{B}$ denote a Boolean σ-algebra and let $\mathfrak{M}$ be the measure ring of a probability space $P = [U, \mathscr{U}, u]$. Then a σ-algebra homomorphism F of $\mathscr{B}$ into $\mathfrak{M}$ is called a "$\mathscr{B}$

measurable generalized random variable" (abbreviated G.R.V.) on P. If $\mathscr{B}$ is a σ-field of subsets of a set S, then F is said to be "S-valued". In most of the discussion, S is the dual of a real topological vector space B and $\mathscr{B}$ is the σ-field of "weak star measurable subsets of S" defined as follows: For $x \in B$ and $\varphi \in S$, let (x, φ) denote the value of φ on x and let $(x, \cdot)$ denote the linear functional on S determined by x. Then $\mathscr{B}$ is the least σ-field of subsets of S such that $(x, \cdot)$ is measurable relative to $\mathscr{B}$ for every $x \in B$.

In the above situation, the author introduces a notion of integration for certain G.R.V.'s which agrees with the weak integral of Pettis [same Trans. **44** (1938), 277–304] when B is a Banach space and the G.R.V. is induced by a point function f on U to S (i.e. $F(b)=f^{-1}(b)$, $b \in \mathscr{B}$). The indefinite integral $I(M)$ is defined on an ideal in $\mathfrak{M}$ and has values in S. It is countably additive relative to convergence in the weak star-topology of S and is absolutely continuous with respect to the measure u in the following sense: There exists a collection M_α of elements of $\mathfrak{M}$, whose supremum is the unit of $\mathfrak{M}$, and a collection K_α of weak star-compact subsets of S such that $M \in \mathfrak{M}$ and $M \leq M_\alpha$ imply that $I(M)$ is defined and $I(M) \in u(M)K_\alpha$. The fundamental result of the paper is the following Radon-Nikodym Theorem: Let v be a function defined on $\mathfrak{M}$ to S which is both weak star countably additive and absolutely continuous (in the above sense) with respect to u. Then v is the indefinite integral of a unique S-valued $\mathscr{B}$ measurable G.R.V. on P. If B is metrizable and separable, then the G.R.V. is induced by a point function. A number of applications of this theorem are given along with various other results for G.R.V.'s. *C. E. Rickart.*

See also: Ul'yanov, p. 22; Fleming and Young, p. 43; Špaček, p. 45; Rényi, p. 69.

Functions of Complex Variables

★ **Knopp, Konrad. Funktionentheorie. I. Grundlagen der allgemeinen Theorie der analytischen Funktionen.** Neunte, neubearbeitete Auflage. Sammlung Göschen Bd. 668. Walter de Gruyter and Co., Berlin, 1957. 144 pp. DM 2.40.

The ninth edition of this well-known work is changed only in details.

Jankowski, W. Sur les zéros d'un polynôme contenant un paramètre arbitraire. Ann. Polon. Math. **3** (1957), 304–311.

Let $P(z)$ be a polynomial of degree p with all its zeros in the circle $|z-c_1| \leq r_1$ and let $Q(z)$ be a polynomial of degree q $(q>p)$ with all its zeros in the circle $|z-c_2| \leq r_2$. The author determines the radius $R=R(c_0)$ of a circle $|z-c_0| \leq R$ which contains at least p zeros of every polynomial $F(z)=P(z)+aQ(z)$, where a is an arbitrary complex constant. He finds that $R=\max(A, B)$, where

$$A=(q-p)^{-1}\{pr_2+qr_1+p|c_2-c_0|+q|c_1-c_0|\},$$
$$B=r_2+|c_2-c_0|.$$

For certain cases he proves further that the limits are attained. These results generalize those obtained in the special case $c_1=c_2=0$ by M. Biernacki [Bull. Internat. Acad. Polon. Sci. Lett. Cl. Sci. Math. Nat. Sér. A. **1927**, 541–685], but use essentially the same methods as Biernacki's. *M. Marden* (Milwaukee, Wis.).

Springer, George. On Morera's theorem. Amer. Math. Monthly **64** (1957), 323–331.

A theorem is proved which is somewhat more general than the following corollary. Let f be a continuous, single-valued function in a region R of the complex plane and let $\{U_i\}$ be an arbitrary open covering of R. If $\int_C f(z)\,dz=0$ for every circle C that lies entirely within at least one open set of the covering $\{U_i\}$, then f is holomorphic in R.

Frame, J. S. Power series expansions for inverse functions. Amer. Math. Monthly **64** (1957), 236–240.

The author derives explicit formulas for the coefficients of the power series of the inverse function directly in terms of the coefficients of the power series of the original function. His results are closely related to the well-known Lagrange formula for the reversion of series. The paper then shows how the above-mentioned formula can be applied to the solution of the trinomial equation $z^n+az^m+b=0$. *F. Herzog.*

Artémiadis, Nicolas K. Sur les coefficients de Taylor de certaines classes de fonctions. C. R. Acad. Sci. Paris **244** (1957), 713–715.

If $F(z)=1+\sum_{n=1}^{\infty} a_n z^n$ is regular in $|z|<1$ and, for any $\alpha>0$, there is an $r_0>0$ such that $\Re F(z) \geq \Im F(z) \tan \frac{1}{2}\alpha$ for $1>r\geq r_0$, then $|a_n+a_{n+1}| \leq 2$ and $|a_n| \leq 2n+1$. Further, if $F(r) \sim 1/(1-r)$ $(r \uparrow 1)$, then $a_1+a_2+\cdots+a_{n-1}+a_n/2 \sim n$. On the other hand, if $F(z)=z+\sum_{n=2}^{\infty} a_n z^n$ is regular in $|z|<1$ and, for any $\alpha>0$, there is an $r_0>0$ such that $\Re F(z)+[(1+r)/(1-r)]\Im F(z) \tan \frac{1}{2}\alpha \geq 0$ $(r_0<r<1)$, then $|a_{n+2}-a_n| \leq 2$. Further, if $F(r) \sim 1/(1-r)$ $(r \uparrow 1)$, then $\sum_{\nu=1}^{n} \nu^{-1} \sum_{\mu=2}^{\nu} a\mu \sim n$. The proofs use theorems by the author [same C. R. **244** (1957), 544–547; MR **18**, 575] and Rogosinski [Math. Z. **35** (1932), 93–121]. *S. Izumi.*

Bagemihl, F. On power series with unbounded cluster sets, and functions of class H_2 with meager sets of radial continuity. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 763–765.

The author proves an analogue to the theorem of Ryll-Nardzewski and Steinhaus on singularities of Taylor series [Studia Math. **12** (1951), 159–165; MR 13, 732]. The result is the same as the last theorem mentioned in the paper on power series by Gaier and Meyer-König [Jber. Deutsch. Math. Verein. **59** (1956), Abt. 1, 36–48; MR **18**, 385]. The following corollary is of special interest: Let $p>1$, and let X_p denote the Banach space of functions f that belong to the Hardy class H_p; then there exists a residual set R_p in X_p such that, for every f in R_p, the radii of the unit disk on which f is bounded form a set of first category. *G. Piranian* (Ann Arbor, Mich.).

Ul'yanov, P. L. On Cauchy A-integrals for contours. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 383–385. (Russian)

The author has been applying the generalized integral called the A-integral to a variety of problems [see, e.g., same Dokl. **102** (1955), 1077–1080; MR **17**, 137; Moskov. Gos. Univ. Uč. Zap. **181**. Mat. **8** (1956), 139-157; MR **18**, 892]. Here he extends the definition to contour integrals of complex-valued functions, states a theorem on change of variable for A-integrals, and states the following theorems on Cauchy integrals. (1) If G is a region bounded by a curve $z=z(s)$ such that $z'(s)$ satisfies a Hölder condition then the function F defined inside G by the Cauchy integral of an L-integrable function is also the A-Cauchy

integral of its non-tangential boundary values. (2) The A-integral over the boundary of G of the product of the boundary values of F and any bounded analytic function is zero. *R. P. Boas, Jr.* (Evanston, Ill.).

Delange, Hubert. Sur la forme forte du théorème de Cauchy relatif à l'intégrale d'une fonction de variable complexe. Bull. Sci. Math. (2) **80** (1956), 156–160.

Der Cauchy'sche Integralsatz wird zum Beweise der klassischen Resultate der komplexen Analysis nur in der schwachen Form benötigt: Ist C eine Jordankurve der z-Ebene und ist $f(z)$ auf C und im Innern I von C holomorph, so gilt $\int_C f(z)dz=0$. Bekanntlich ist aber auch noch eine stärkere Form richtig: Schon dann, wenn $f(z)$ in I holomorph und auf C nur stetig ist, folgt $\int_C f(z)dz=0$. Der Verf. gibt für diesen Satz unter Verwendung des Riemannschen Abbildungssatzes einen einfachen Beweis.
H. Grauert (Münster).

Gahov, F. D. On the inverse boundary problem of a multiply connected domain. Rostov. Gos. Ped. Inst. Uč. Zap. no. 3 (1955), 19–27. (Russian)

Let (1) $\{u_k(s), v_k(s)\}$ $(k=0, 1, \cdots, n)$ denote $n+1$ pairs of continuous periodic functions with respective periods l_k. Let L_{wk} denote the family of $n+1$ curves which (1) determines in the w-plane; suppose that each of the curves is simple, that no two of them intersect, and that the family of curves determines a finite domain D_w of connectivity $n+1$.

The inverse boundary problem is this: to construct $n+1$ rectifiable closed curves L_{zk} which define a finite domain D_z (not necessarily single-sheeted) in the z-plane such that the complex-valued functions

$$w_k(s)=u_k(s)+iv_k(s),$$

with s representing arc length on L_{zk}, give the boundary values of a function $w(z)$ which effects the conformal mapping of D_z onto D_w. Under the hypothesis that the u_k and v_k are differentiable, the author obtains some necessary and some sufficient conditions on (1) for the existence of a solution of the problem. *G. Piranian.*

Egorov, P. M. Application of the method of conformal transformations to the modelling of three-dimensional potential and vortical fields. Električestvo **1956**, no. 5, 31–38. (Russian)

Călugăreanu, G. Sur les domaines univalents. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 853–860. (Romanian. Russian and French summaries)

Let D be a Riemannian domain with multiple but at most finite p-fold covering. Let D_q denote the maximal open set of points of the plane which are covered exactly q times $(q=1, \cdots, p)$. Let A be the measure of D, assumed finite, and let A_q be the measure of D_q; thus $A=\sum_{q=1}^{p} qA_q$. The author aims at setting up analytic conditions which will assure that D is univalent, or $p=1$. Let

$$g(\lambda)=\frac{1}{\pi\Gamma((\alpha+2)/\alpha)}\iint_{DD}\lambda^{2/\alpha}\exp\left(-\gamma r_{12}^{\alpha}\right)d\omega_1d\omega_2,$$

where α and λ are positive parameters, r_{12} is the distance between the points p_1, p_2 of D and $d\omega_i$ denotes the area element at P_i. The author proves that

$$\lim_{\lambda\to\infty} g(\lambda)=\sum_{q=1}^{p} q^2A_q. \tag{1}$$

The function $g(\lambda)$ is also shown to be increasing for sufficiently large values of λ. Now if $p=1$, then $g(\lambda)\to A_1=A$, while if $p>1$ we obtain by (1) that

$$g(\lambda)\to\sum q^2A_q>\sum qA_q=A.$$

Thus $\lim g(\lambda)=A$, or $g(\lambda)<A$ for sufficiently large values of λ, is a necessary and sufficient condition for the univalency of D. This idea is further elaborated by expressing the condition in terms of the so-called mean diameters of D:

$$\delta_n=\left(A^{-2}\iint_{DD} r_{12}^{\,n}d\omega_1d\omega_2\right)^{1/n},$$

and also in other ways, one of which leads to a characteristic set of explicit finite inequalities. For these further details we must refer to the paper. *I. J. Schoenberg.*

Cazacu, Cabiria Andreian. Sur la classe des surfaces de Riemann normalement exhaustibles et ses relations avec d'autres classes. C. R. Acad. Sci. Paris **242** (1956), 2281–2283.

Relations between various classes of covering surfaces are stated: normally exhaustible, A_∞, regularly exhaustible, the (I) surfaces of Stoilow. The details appear in Math. Nachr. **15** (1956), 77–86 [MR **18**, 647].
M. H. Heins (Providence, R.I.).

Kuramochi, Zenjiro. Mass distributions on the ideal boundaries of abstract Riemann surfaces. II. Osaka Math. J. **8** (1956), 145–186.

[For part I see same J. **8** (1956), 119–137; MR **18**, 120.] Detailed account of the results summarized in Proc. Japan. Acad. **32** (1956), 228–233, 234–236 [MR **18**, 290].
M. H. Heins (Providence, R.I.).

Lambin, N. V. Lines of symmetry of non-simply-connected Riemann surfaces. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 84–85. (Russian)

Leont'ev, A. F. On a question of interpolation in the class of entire functions of finite order. Mat. Sb. N.S. **41**(83) (1957), 81–96. (Russian)

The problem is whether there is an entire function of growth not exceeding order ρ, finite type, taking prescribed values a_n at prescribed points λ_n under the restriction $|a_n|<\exp(K|\lambda_n|^\rho)$. The author has solved it for positive λ_n [Dokl. Akad. Nauk SSSR (N.S.) **66** (1949), 331–334; MR **10**, 695], but his conditions were not necessary in the general case. Here the obtains the following necessary and sufficient conditions: (1) $\lim\sup n|\lambda_n|^{-\rho}<\infty$ (2) $\lim\sup|\lambda_n|^{-\rho}\log|1/\eta_n|<\infty$, where $\eta_n=\prod(1-\lambda_n/\lambda_s)$, the product extending over all $\lambda_s\neq\lambda_n$ for which $|\lambda_s|/|\lambda_n|$ is between two fixed positive numbers.
R. P. Boas, Jr. (Evanston, Ill.).

Stein, E. M. Functions of exponential type. Ann. of Math. (2) **65** (1957), 582–592.

The paper contains three independent parts, one of which is not concerned with functions of exponential type; the parts are connected by the use of convolution arguments to extend L^∞ results to L^p results and one-variable results to several-variable results. Part 1 contains a general inequality of which a special case is the L^p-form of Bernstein's inequality for the derivative of an entire function of exponential type; the method yields a determination of the case of equality in the theorem (a new

result). Part 2 replaces L^∞ norms by L^p norms in Kolmogorov's inequality connecting the bounds of three derivatives of a function on the real axis [cf. Amer. Math. Soc. Transl. no. 4 (1949); MR 11, 86]; as an application the author shows that if f and $f^{(n)} \in L^p$ ($1 \leq p < \infty$) then $f^{(k)} \in L^p$ ($0<k<n$). Part 3 contains the following generalization of the Paley-Wiener theorem. Let E_n be Euclidean n-space, $f \in L^2(E_n)$, f bounded, F the Fourier transform of f. Let Ω be a compact, convex, symmetric domain, let Ω^* consist of points x such that $|x \cdot y| \leq 1$ if $y \in \Omega$, let $\psi(x)$ be the characteristic function of Ω^* and let

$$U_t(f)(x) = \int t^{-n}\psi(z/t)f(x+z)dz,$$

with integration over E_n. Then F vanishes outside Ω if and only if $U_t(f)(x)$ is, for each fixed x, an entire function of t of exponential type at most 1. *R. P. Boas, Jr.*

Boas, R. P., Jr. Inequalities for functions of exponential type. Math. Scand. 4 (1956), 29–32.

Let $f(z)$ be an entire function of exponential type τ with $|f(z)| \leq M$ on the real axis. An interpolation formula is derived which yields the following inequality

$$|f(x+iy)e^{-i\omega}+f(x-iy)e^{i\omega}| \leq 2M[\cosh^2 \tau y - \sin^2 \omega]^{\frac{1}{2}},$$

where ω is an arbitrary real constant. This includes various other inequalities. In particular, if $f(z)$ is real on the real axis, then (a) $|f(x+iy)| \leq M \cosh \tau y$ and (b) $|\mathrm{Im} f(x+iy)| \leq M \sinh \tau |y|$. Here (a) is due to Duffin and Schaeffer [Bull. Amer. Math. Soc. 44 (1938), 236–240] and (b) is due to Hörmander [Math. Scand. 3 (1955), 21–27; MR 17, 247]. The interpolation formula also yields inequalities on the L_p norm of $f(z)$ on lines parallel to the real axis. *R. J. Duffin.*

Hiong, King-Lai. Sur les fonctions algébroïdes et leurs dérivées. Etude des défauts absolus et des défauts relatifs. Ann. Sci. Ecole Norm. Sup. (3) 73 (1956), 439–451.

The function $u(x)$ is said to be algebroid for $|x| < R(\leq \infty)$ (with ν branches) if it satisfies an equation

$$\psi(u) \equiv A_\nu(x)u^\nu + A_{\nu-1}(x)u^{\nu-1} + \cdots + A_0(x) = 0,$$

where (i) the coefficients A_j are holomorphic for $|x|<R$ and (ii) there exists no value of x ($|x|<R$) for which all A_j vanish. Let $\psi_1(u') \equiv B_\nu(u')^\nu + B_{\nu-1}(u')^{\nu-1} + \cdots + B_0 = 0$. The fundamental results of Nevanlinna's theory of meromorphic functions have been extended to algebroid functions by H. L. Selberg and Valiron. The author extends to algebroid functions various inequalities of the same general character as Nevanlinna's second fundamental theorem. A typical example is the following analogue of an inequality due to Milloux [same Ann. (3) 63 (1947), 289–316; MR 9, 342]: Let $a_1, a_2, \cdots, a_p$ be finite distinct numbers and $b_1, b_2, \cdots, b_q$ distinct but not necessarily finite; furthermore no b_j is zero. Then

$$\nu p(q+1-2\nu)T(r, u) < (q+1-2\nu) \sum_{k=1}^{p} N\left(r, \frac{1}{\psi(a_k)}\right) + \sum_{i=1}^{q} N\left(r, \frac{1}{\psi_1(b_i)}\right) - (q-2\nu)N\left(r, \frac{1}{\psi_1(0)}\right) + S_1(r,u),$$

where $S_1(r, u)$ is, in general, negligible with respect to $T(r, u)$.

The author also shows that the inequalities contained in his paper imply various relations between the values of the deficiencies of u and u'. *A. Edrei.*

Schubart, Hans; und Wittich, Hans. Über die Lösungen der beiden ersten Painlevéschen Differentialgleichungen. Math. Z. 66 (1957), 364–370.

All solutions of the first two differential equations of Painlevé

$$w'' = 6w^2 + a(z), \quad a(z) = A_0 z + A,$$
$$w'' = 2w^3 + b(z)w + C, \; b(z) = B_0 z + B,$$

are single-valued and meromorphic for $z \neq \infty$. The special case $A_0 = B_0 = 0$ can be integrated with the help of elliptic functions and their degenerates.

Applying Nevanlinna's theory and making use of their previous results [see especially Wittich, Neuere Untersuchungen über eindeutige analytische Funktionen, Springer, Berlin, 1955; MR 17, 1067], the authors study the value distribution properties of the solutions. It is first proved that the solutions possess a completely ramified value c, in the sense that $w(z)-c=0$ has only multiple roots, if and only if $A_0 = B_0 = 0$. For the latter equation, the total index of ramification Φ_e for finite values is always equal to its largest possible value 2 if $C \neq 0$. [The standard notations are used; cf. Nevanlinna, Eindeutige analytische Funktionen, 2nd ed., Springer, Berlin, 1953; MR 15, 208.] This occurs although the ramification index may vanish for all individual values, as is shown by an example. For the first equation, $\vartheta(\infty) + \Phi_e = \frac{1}{2} + \frac{3}{2} = 2$. From the results obtained it also appears that in all cases the second main theorem holds for the solutions as an asymptotic equality

$$N_1(r) + \sum_{1}^{q} m(r, c_j) = 2T(r, w) + O(\log r).$$

O. Lehto (Helsinki).

Pogorzelski, W. Problème non linéaire d'Hilbert pour le système de fonctions. Ann. Polon. Math. 2 (1955), 1–13.

Let S_ν^-, ($\nu = 1, 2, \cdots, p$), be a set of disjoint domains in the complex plane, bounded by non-intersecting closed Jordan curves L_ν, ($\nu = 1, 2, \cdots, p$), respectively. Let S^-_0 be the infinite domain lying outside a closed Jordan curve L_0 which surrounds all the L_ν, ($\nu = 1, 2, \cdots, p$), and let S^+ be the domain lying between L_0 and the L_ν ($\nu = 1, 2, \cdots, p$). The author shows how to find the system of functions $\Phi_\mu(z)$, ($\mu = 1, 2, \cdots, m$), each holomorphic in the domains S^+, S_ν^-, ($\nu = 0, 1, \cdots, p$), separately, and whose limit values $\Phi_\mu^+(t)$, $\Phi_\mu^-(t)$ concerning the domains S^+, S_ν^- satisfy for all t on $L = \sum_{\nu=0}^{p} L_\nu$ the relations

$$(1) \quad \Phi_\mu^+(t) = G_\mu(t)\Phi_\mu^-(t) + \lambda F_\mu[t, \Phi_1^+(t), \cdots, \Phi_m^+(t), \Phi_1^-(t), \cdots, \Phi_m^-(t)],$$

($\mu = 1, 2, \cdots, m$), where the $G_\mu(t)$ are given functions on the lines L, L_ν, the $F_\mu(t, u_1, u_2, \cdots, u_{2m})$ are given functions of $(2m+1)$ variables, and λ is a parameter. The solution is derived from the known solution

$$\Phi_1(z) = \frac{\lambda}{2\pi i} X_1(z) \int_L \frac{F_1(t)dt}{X_1(t)(t-z)} + X_1(z)P_1(z)$$

of the linear Hilbert problem obtained from (1) above by putting $\mu = 1$, $m = 0$; here $P_1(z)$ is an arbitrary integral function and $X_1(z)$ is a canonical solution of the homogeneous Hilbert problem. For his problem the author requires the following conditions, which are more severe than those needed for the solution of the linear problem: the lines L_ν, ($\nu = 0, 1, 2, \cdots, p$), can be enclosed in bands Λ_ν,

($\nu=0, 1, 2, \cdots, p$), respectively, having no common points; in each of these bands $G_\mu(z)$, ($\mu=1, 2, \cdots, m$), are holomorphic, and so are $F_\mu(z, u_1, u_2, \cdots, u_m, u_{m+1}, \cdots, u_{2m})$, ($\mu=1, 2, \cdots, m$), considered as functions of z;

$$G_\mu(t)\neq 0 \text{ on } L_\nu, \ (\nu=0, 1, 2, \cdots, p);$$

$F_\mu(z, u_1, u_2, \cdots, u_m, u_{m+1}, \cdots, u_{2m})$ ($\mu=1, 2, \cdots, m$), are holomorphic with respect to each u_q ($q=1, 2, \cdots, 2m$), in the domain $|u_q|\leqq R$ ($q=1, 2, \cdots, 2m$).

N. A. Bowen (Zbl. **65** (1956), 64).

Dundučenko, L. E. Certain extremal properties of analytic functions given in a circle and in a circular ring. Ukrain. Mat. Ž. **8** (1956), 377–395. (Russian)

The author examines various classes of functions of the form $w(z)=z^p+\sum_{n=1}^{\infty} c_{n+1}z^{p+n}$ (p a positive integer) which are regular in the unit circle. These are (A) the class of p-valent convex functions, (B) the class p-valent starlike functions, (C) the class of univalent starlike and bounded functions, and (D) a special class of univalent functions introduced by Rahmanov [Dokl. Akad. Nauk SSSR (N.S.) **91** (1953), 729–732; MR **15**, 413]. For the class (A) the author obtains sharp upper and lower bounds for $|w'(z)|$ and $|w(z)|$ and sharp upper bounds for $|\arg w'(z)|$ and $|c_n|$. Similar results are obtained for the class (B). For the class (C) sharp upper and lower bounds are obtained for $|w'(z)|$ and $|w(z)|$ and sharp upper bounds for $|c_2|$ and $|c_3|$. For the class (D) the author proves that $|w'(z)|\leqq 1/(1-|z|)^2$ and that $|c_n|\leqq 1$ ($n=2, 3, \cdots$). Let $K_r(\alpha)$ be the curvature at the image of $z_0=re^{i\alpha}$ of the curve which is the image of $|z|=r$ under the mapping $w(z)$. The author obtains sharp upper and lower bounds for $K_r(\alpha)$ for the class (A).

The author extends many of these results to functions regular in the ring $q<|z|<1$.

The author is apparently not aware of the fact that many of his results were obtained earlier by the reviewer [Trans. Amer. Math. Soc. **68** (1950), 204–223; MR **11**, 508] and by M. S. Robertson [Ann. of Math. (2) **38** (1937), 770–783; Duke Math. J. **12** (1945), 669–684; MR **7**, 379].

A. W. Goodman (Lexington, Ky.).

Jenkins, James A. On a conjecture of Spencer. Ann. of Math. (2) **65** (1957), 405–410.

Let $w=f(z)$ be regular for $|z|<1$ with $f(0)=0$, $f'(0)=1$; and let it map $|z|<1$ onto a Riemann surface $\mathfrak{F}$ over the w-plane. $f(z)$ is said to be logarithmically areally mean 1-valent if $\mathfrak{F}$ covers some circle $|w|<R_0$ simply while the portion of $\mathfrak{F}$ above every annulus $R_0<|w|<R$ has logarithmic area [metric $(2\pi)^{-1}|w|^{-1}|dw|$] at most $2\pi \log R/R_0$. The following conjecture of Spencer [Ann. of Math. (2) **42** (1941), 614–633; MR **3**, 78], which generalizes a classical result on schlicht functions, has recently been proved by Garabedian and Royden [ibid. **59** (1954), 316–324; MR **15**, 613]: $f(z)$ takes every value w with $|w|<\frac{1}{4}$ in $|z|<1$; a w with $|w|=\frac{1}{4}$ is omitted only when $f(z)=z/(1+\epsilon z)^2$ $|\epsilon|=1$. The author gives here a new proof based on the method of an extreme metric and circular symmetrizations as used by him, and also by Spencer, in earlier papers on this topic.

W. W. Rogosinski.

★ **Perron, Oskar. Die Lehre von den Kettenbrüchen. Dritte, verbesserte und erweiterte Aufl. Bd. II. Analytisch-funktionentheoretische Kettenbrüche.** B. G. Teubner Verlagsgesellschaft, Stuttgart, 1957. vi+316 pp. DM 49.00.

This is volume II of the third edition of this book, the first edition of which appeared in 1913 and the second edition with a few changes in 1929 (Leipzig). The present volume, as was the case with volume I [1954; MR **16**, 239], has been very materially changed and brought up to date. It deals entirely with the analytic, function-theoretic aspect of continued fraction theory, whereas volume I deals entirely with the elementary arithmetic part.

Comparison with earlier editions shows the following improvements and additions: In Chapter I, section 1 has been added as it contains a recapitulation of the essential recurrence relations developed in volume I. In section 3 are derived transformations of continued fractions into others whose sequences of approximants are permutations of the original ones, and also continued fractions with prescribed approximants. Section 7 on the transformation of Bauer and Muir has been considerably altered in its illustrative material. Section 8 on further applications and the principal formula of Ramanujan is entirely new. Chapter 2 contains many additions on new criteria for the convergence and divergence of continued fractions, since a great deal of new work in this field has been done recently. In section 10 the criterion of Scott-Wall has been added. Section 15 on the convergence criteria of van Vleck-Jensen and Hamburger-Mall-Wall has been changed and brought up to date. Section 16 on the region of validity of the Ramanujan formula is new, as well as section 17 on the parabola theorem. Chapter 3 begins with a new section on general C-fractions. In sections 20 and 24 are given more remarkable formulas of Ramanujan proved by G. N. Watson. Other new sections are 30, which is concerned with J-fractions and their applications to polynomials whose zeros have negative real parts, 31 on new developments in further types of continued fractions corresponding to power series, including Schur fractions and extensions thereof, and 38 on the convergence of G-fractions with developments of Hamburger, Hellinger, and Wall. Section 39 on the moment problem has also been changed. There is a bibliography of 185 references, all of which are cited in the text.

There have been some omissions from the 1929 edition to make room for the new additions. Of course, not all new theorems are included. As the author states, in the case of the new convergence criteria, he has included those which are the most useful and make a comprehensive, well-rounded presentation of the subjet. In all, he has given a fine account of the theory of continued fractions, all of which is of both practical and theoretical interest.

E. Frank (Evanston, Ill.).

Temlyakov, A. A. Integral representation of functions of two complex variables. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 89–92. (Russian)

A circular domain D in the complex $(w; z)$-space is bounded by the hypersurface $|w|=r_1(\tau)$,

$$|z|=\exp(-\int_0^\tau \tau(1-\tau)^{-1}d \ln r_1(\tau)=r_2(\tau) \quad (0\leqq\tau\leqq 1),$$

where $r_1(0)=0$, $0<r_1'(\tau)\leqq\tau^{-1}r_1(\tau)$, $r_1(1)<\infty$. A function $F(w; z)$ is analytic in D and together with its first order partial derivatives continuous in the closure of D. The author proves the integral representation

$$F(w; z)=(4\pi^2 i)^{-1}\int_0^{2\pi} dt\int_0^1 d\tau\int_C (\zeta-u)^{-1}\Phi(r_1(\tau)\zeta, r_2(\tau)\eta)d\zeta.$$

where $\eta=\zeta e^{-it}$, $u=\tau(r_1(\tau))^{-1}w+(1-\tau)(r_2(\tau))^{-1}ze^{-it}$ while Φ is the function $F+wF_w'+zF_z'$ and C is the circle

$|\zeta|=1$. The integral gives the values of F inside D expressed by the values of the function $F+wF_w'+zF_z'$ on the boundary of D. *H. Tornehave* (Copenhagen).

Eremine, S. A. Sur des fonctions entières de deux variables. Ukrain. Mat. Ž. 9 (1957), 30–43. (Russian. French summary)

Let $\varphi(w,z)=\sum_{m,n=0}^{\infty} c_{mn}w^m z^n$ be an entire function of order ρ and type σ. The author proves that $\varphi(w,z)$ is identically zero if $[\rho,\sigma]$ is at most $[1, 2\ln 2]$ and $\varphi(w,z)$ satisfies a set of relations

$$\left(\frac{\partial^{m+n}\varphi(w,z)}{\partial w^m \partial z^n}\right)_{\substack{w=w_m\\ z=z_n}}=0\ (|w_n|<1, |z_n|<1;\ m,n=0,1,2,\cdots).$$

The second theorem of the paper states that every entire transcendental function with $[\rho,\sigma]$ at most $[1, 2\ln 2]$ possesses a sequence of partial derivatives, which are simple (one-sheeted) in the unit bicylinder. The remaining part of the paper deals with two sequences w_m, z_n; $m, n=0, 1, 2, \cdots$, both tending to infinity. With the functions $Q_m(w)=(w-w_0)\cdots(w-w_{m-1})$;

$$T_n(z)=(z-z_0)\cdots(z-z_{n-1})$$

the Newton interpolation sums $\sum_{j,k=0}^{m,n} a_{jk}Q_j(w)T_k(z)$ are constructed and the author proves four more theorems giving sufficient conditions for the convergence of the sequence of Newton sums or of a subsequence of this sequence. There are many misprints and some oddly inconvenient notations which leads to some puzzling relations as, e.g., the following:

$$\max_{|w|=|z|=r} |\varphi(w,z)|<r^{2N}\sum_{m=0}^{T_{mn}(r)}\sum_{m=0}^{N}|c_{mn}|+ 2|T_{mn}(r)+1|^2 \max_{m\leq 0, n\leq 0}|c_{mn}|r^{m+n}+8.$$

H. Tornehave (Copenhagen).

Elianu, I. P. La dérivée aréolaire et la différentielle extérieure. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 8 (1956), 39–50. (Romanian. Russian and French summaries)

Let $f(z^1,\cdots,z^n)$ be a complex-valued (not necessarily analytic) function of the complex variables z^r $(r=1,\cdots,n)$ defined in a region D_{2n} of $2n$-dimensional Euclidean space with coordinates $x^1, y^1, x^2, \cdots, y^n$ $(z^r=x^r+iy^r)$. Using z^r and $\bar z^r$ $(r=1,\cdots,n)$ as independent variables, the author points out that the exterior derivative of the exterior differential form $[f\,dz^1\cdots dz^n]$ is equal to

$$\frac{\partial f}{\partial \bar z^r}[d\bar z^r dz^1\cdots dz^n]$$

(summation convention used), and deduces that f is an analytic function of $z^1,\cdots,z^n$ if and only if $[f\,dz^1\cdots dz^n]$ is closed. He discusses the corresponding results for the real and imaginary parts of the differential forms concerned, and some developments therefrom.

A. Erdélyi (Pasadena, Calif.).

See also: Babenko, p. 20; Teleman, p. 61; Wu, p. 8.6

Geometric Analysis

See: Elianu, p. 26; Schouten, p. 59.

Harmonic Functions, Convex Functions

Freud, G.; und Králik, D. Über die Anwendbarkeit des Dirichletschen Prinzips für den Kreis. Acta Math. Acad. Sci. Hungar. 7 (1956), 411–418. (Russian summary)

This paper contains a discussion of the relation to each other of certain previously known conditions for the applicability of Dirichlet's principle to the solution of the Dirichlet problem for the circle. Some refinements of these conditions are also given. Among the previous conditions involved is that due to R. Courant, in terms of the Fourier coefficients of the boundary value function [Dirichlet's principle, conformal mapping and minimal surfaces, Interscience, New York, 1950; MR 12, 90]. Other results cited include those due to S. M. Nikol'skiĭ [Mat. Sb. N.S. 35(77) (1954), 247–266; MR 16, 589], and L. N. Slobodeckiĭ and V. M. Babič [Dokl. Akad. Nauk SSSR (N.S.) 106 (1956), 604–606; MR 17, 959].

F. W. Perkins (Hanover, N.H.).

Kalicin, Nikola St. On some formulas for biharmonic functions. Univ. d'Etat Staline Fac. Tech. Constructions. Annuaire 5 (1949–1950), 197–224. (Bulgarian. Russian summary)

Huber, Alfred. On the reflection principle for polyharmonic functions. Comm. Pure Appl. Math. 9 (1956), 471–478.

Let $w(x_1, x_2, \cdots, x_n)$ be harmonic of order p (that is $\Delta^p w=0$) in a region G whose boundary contains an open subset S of the hyperplane $x_1=0$. If w/x_1^{p-1} assumes the boundary value zero on S, it is shown that w can be continued analytically across S into the reflected domain G^* by the relation

$$w^*=\sum_{k=0}^{p-1}(-x_1)^{p+k}(k!)^{-2}\Delta^k(w/x_1^{p-k}).$$

Here w^* denotes $w(-x_1, x_2, \cdots, x_n)$. For $p=1$ this is the classical reflection principle of Schwartz. For $p=2$ the reflection relation was given by Poritsky [Trans. Amer. Math. Soc. 59 (1946), 248–279; MR 7, 449]. The proof of the above theorem, in the case $p=2$, was given by Duffin [Duke Math. J. 22 (1955), 313–324; MR 18, 29] and the proof for arbitrary p is here developed along similar lines. The paper concludes with an application to a growth theorem of Liouville type for polyharmonic functions; this answers a question raised by A. Weinstein.

R. J. Duffin (Pittsburgh, Pa.).

Protter, M. H. A generalization of completely convex functions. Duke Math. J. 24 (1957), 205–213.

A function $f(x)$ is completely convex in (a,b) if $f^{(4k)}(x)\geqq 0$ and $f^{(4k+2)}(x)\leqq 0$ there. The author calls f almost completely convex if $f^{(4k)}(x)\geqq 0$ and

$$f^{(4k+2)}(a)+f^{(4k+2)}(b)\leqq \pi^2(b-a)^{-2}[f^{(4k)}(a)+f^{(4k)}(b)],$$

and shows that such functions share with completely convex functions the property [Widder, The Laplace transform, Princeton, 1941, pp. 178–179; MR 3, 232] of being restrictions of entire functions of exponential type. The generalization is significant, since, for example, e^{x^2} is almost completely convex on (0, 1) but not completely convex. The author also extends Widder's theorem to two variables: here $f(x,y)$ is completely convex if

$$\partial^{2(p+q)}f/\partial x^{2p}\partial y^{2q}\geqq 0$$

or $\leqq 0$ according as $p+q$ is even or odd, and then it is the restriction of an entire function of exponential type. Complete convexity in each variable alone does not suffice.

R. P. Boas, Jr. (Evanston, Ill.).

See also: Kuramochi, p. 23; Friedman, p. 39; Payne and Weinberger, p. 39.

Special Functions

Krall, H. L. Polynomials with the binomial property. Amer. Math. Monthly **64** (1957), 342–343.

A sequence of polynomials $\{b_n(x)\}$ $(n=0, 1, \cdots;$ b_n of degree $n)$ is "binomial" if

$$b_n(x+y)=\sum_{i=0}^{n}\binom{n}{i}b_i(x)b_{n-i}(y)\ (n=0, 1, \cdots).$$

Let $B_n(x)=(n!)^{-1}b_n(x)$. Theorem: $\{b_n(x)\}$ is binomial if and only if $\{B_n(x)\}$ satisfies the relations $B_0(x)=1$; $B_n(0)=0$ $(n>0)$; $\sum_{k=1}^{\infty} c_k B_n^{(k)}(x)=B_{n-1}(x)$ $(n=1, 2, \cdots)$ for some constant sequence $\{c_k\}$ with $c_1\neq 0$. ($B_n^{(k)}$ is the kth derivative of B_n.)

I. M. Sheffer.

Danese, Arthur E. Some inequalities involving Hermite polynomials. Amer. Math. Monthly **64** (1957), 344–346.

Généralisant des résultats de Mukherjee et Nanjundiah [Math. Student **19** (1951), 47–48; MR **13**, 649], l'auteur démontre des inégalités satisfaites par les polynomes d'Hermite, telles que (pour $n\geqq 1$)

$$[H_n(x)]^{2(2r+1)}-[H_{n+1}(x)]^{2r+1}[H_{n-1}(x)]^{2r+1}>0,$$

(r entier positif);

$$n(n+1)H_n^2-n(n-1)H_{n+2}H_{n-2}>0;$$

puis d'autres, faisant intervenir les dérivées, telles que $H_n'^2-H_{n+1}H_{n-1}''>0$ et en particulier les dérivées de la quantité $\Delta_n=H_n^2-H_{n+1}H_{n-1}$.

R. Campbell (Caen).

Hirschman, I. I., Jr. Projections associated with Jacobi polynomials. Proc. Amer. Math. Soc. **8** (1957), 286–290.

Dans une première partie, l'auteur, pour simplifier, expose son résultat sur les polynomes de Legendre; il met en évidence une double structure très remarquable pour ces polynomes: 1°) En tant que fonctions de x, ils sont les fonctions propres d'un opérateur différentiel du second ordre

$$Da(x)=-\frac{d}{dx}\Big[(1-x^2)\frac{d}{dx}a(x)\Big]$$

self-adjoint pour les $a(x)$ bornées du sous-espace de L^2 telles que $Da(x)\in L^2$ (valeurs propres: $n(n+1)$). 2°) En tant que fonctions de n, ils sont les fonctions propres de l'opérateur

$$\Delta\alpha(n)=\frac{(n+1)\alpha(n+1)}{2\surd((n+\frac{3}{2})(n+\frac{1}{2}))}+\frac{n\alpha(n-1)}{2\surd((n+\frac{1}{2})(n-\frac{1}{2}))}$$

self-adjoint sur l^2 (valeurs propres: $-1\leqq x\leqq +1$).

Si $D=\int\lambda dG_\lambda$, on a

$$G_\lambda a(x)=\int_{-1}^{+1}\Big[\sum_{n(n+1)<\lambda}(n+\tfrac{1}{2})P_n(x)P_n(t)\Big]a(t)dt.$$

Pollard a montré [Trans. Amer. Math. Soc. **62** (1947), 387–403; **63** (1948), 355–367; Duke Math. J. **16** (1949), 189–191; MR **9**, 280, 426; **10**, 450] que dans L^p $(1\leqq p<\infty)$, $(-1\leqq x\leqq +1)$, si comme d'habitude $\|a_p\|$ désigne

$$\Big[\int_{-1}^{+1}|a(x)|^p dx\Big]^{1/p},$$

on a $\|G_\lambda(ax)\|_p\leqq A(p)\|a(x)\|_p$ avec $4/3<p<4$, $A(p)$ dépendant seulement de p et non de λ. Posant alorsde même $\|\alpha\|_p=[\sum_0^\infty|\alpha(n)|^p]^{1/p}$ et remarquant que

$$\Gamma_\lambda\alpha(n)=\sum_{m=0}^{\infty}\alpha(m)\int_{-1}^{\lambda}\surd((n+\tfrac{1}{2})(m+\tfrac{1}{2}))P_m(x)P_n(x)dx$$

satisfait à $\Delta=\int\lambda d\Gamma_\lambda$, l'auteur démontre dans cette étude l'inégalité analogue $\|\Gamma_\lambda\alpha(n)\|_p\leqq B(p)\|\alpha\|_p$, $B(p)$ dépendant seulement de p et non de λ.

Dans la seconde partie de l'étude, une extension de l'inégalité précédente est obtenue pour les polynomes de Jacobi $P_n^{\sigma\tau}$: $\|\Gamma_\lambda\alpha(n)\|_p\leqq B(\sigma, \tau, p)\|\alpha(n)\|_p$.

R. Campbell (Caen).

Carlitz, L. A note on the Bessel polynomials. Duke Math. J. **24** (1957), 151–162.

Après avoir rappelé que Krall et Frink ont introduit les polynomes y_n,

$$y_n(x)=\sum_{r=0}^{n}\frac{(n+r)!}{(n-r)!r!}\Big(\frac{x}{2}\Big)^r,$$

solutions de l'équation différentielle

$$x^2y''+(2x+2)y'-n(n+1)y=0,$$

et posé $\theta_n(x)=x^n y_n(1/x)$, l'auteur établit des formules relatives [Trans. Amer. Math. Soc. **65** (1949), 100–115; MR **10**, 453] aux θ_n et relie ces fonctions aux polynomes de Laguerre par l'équation

$$(-2)^n\theta_n\Big(\frac{x}{2}\Big)=n!L_n^{(-2n-1)}(x).$$

Il en déduit des formules telles que de transformation du produit de 2 fonctions $\theta(x)$ en somme et d'autres propriétés des polynomes θ_n (inégalités, relations avec les dérivées, et équations aux différences finies).

R. Campbell (Caen).

Carlitz, Leonard. Some polynomials of Touchard connected with the Bernoulli numbers. Canad. J. Math. **9** (1957), 188–190.

Polynomials $Q_n(x)$ defined by Touchard [same J. **8** (1956), 305–320; MR **18**, 16] by $Q_0(x)=1$, $Q_1(x)=2x+1$,

$$Q_{n+1}(x)=(2x+1)Q_n(x)+\frac{n^2}{4n^2-1}Q_{n-1}(x)$$

were determined explicitly by Wyman and Moser [ibid. **8** (1956), 321–322; MR **18**, 17] in the form

$$Q_n(x)=2^n n!\binom{2n}{n}^{-1}\sum_{2r\leq n}\binom{2x+n-2r}{n-2r}\binom{x}{r}^2.$$

These polynomials are here related to polynomials $F_n(x)$ studied by Bateman. Thus, if

$$F_n(x)={}_3F_2\Big[\begin{matrix}-n,\ n+1,\ \frac{1}{2}(1+x)\\ 1\ ,\ 1\end{matrix}\Big],\quad F_n(-x)=(-1)^nF_n(x),$$

then

$$Q_n(x)=(-1)^n2^nn!\binom{2n}{n}^{-1}F_n(2x+1).$$

This leads to a verification of a symbolic orthogonality property of $Q_n(x)$ first proved by Touchard. *L. Moser.*

Brafman, Fred. On Touchard polynomials. Canad. J. Math. **9** (1957), 191–193.

For the polynomials $Q_n(x)$ of the preceding review an alternative expression and several generating functions are given. For example.

$$e^t[L_x(-t)]^2=\sum_{n=0}^{\infty}\frac{Q_n(x)(\frac{1}{2})_n(2t)^n}{(n!)^3}.$$

L. Moser (Edmonton, Alta.).

Brafman, Fred. Some generating functions for Laguerre and Hermite polynomials. Canad. J. Math. **9** (1957), 180–187.

The author derives eleven generating functions that relate the Laguerre and Hermite polynomials to the generalized hypergeometric functions. A sample formula is

$$\exp(2xt-t^2)(1+ut^2)^{-a}{}_1F_1\left[a;\tfrac{1}{2};\frac{ut^2(x-t)^2}{1+ut^2}\right]=\sum_{n=0}^{\infty}{}_3F_1[-\tfrac{1}{2}n,-\tfrac{1}{2}n+\tfrac{1}{2},a;\tfrac{1}{2};u]\frac{H_n(x)t^n}{n!}.$$

M. Wyman (Edmonton, Alta.).

Singh, Vikramaditya. On the orthogonal sub-set of Appell polynomials. Proc. Nat. Inst. Sci. India. Part A. **22** (1956), 26–31.

Appell polynomial sets $\{P_n(x)\}$, defined by the relation $A(t)e^{tx}=\sum_0^\infty t^nP_n(x)$, are considered, the problem being to find the orthogonal subset of the polynomials so defined. Using an integral representation in terms of hypergeometric polynomials [Singh, Proc. Nat. Inst. Sci. India **20** (1954), 341–347; MR **16**, 128],

$$P_n(x)=\frac{x^n}{n!}\int_0^\infty {}_{r+1}F_r\left\{\begin{matrix}-n_1a_1,\cdots,a_r\\ b_1,\ \cdots,b_r\end{matrix};\frac{-t}{x}\right\}d\beta(t)$$

and requiring $\{P_n(x)\}$ to be a sequence of orthogonal polynomials, the problem is reduced to a differential equation for $\beta(t)$ (assumed sufficiently differentiable). The solution again shows that the only orthogonal subset of the Appell set of polynomials is the set of Hermite polynomials. Several infinite series are summed in terms of $A(t)$, using the above mentioned integral representations.

G. E. Latta.

Webster, M. S. Non-linear recurrence relations for certain classical functions. Amer. Math. Monthly **64** (1957), 249–252.

It is shown that the ultraspherical, Hermite and Laguerre polynomials and the Bessel functions can be characterized by means of certain non-linear recurrence relations. For example: Theorem 1. If $f_n(x)$ is a polynomial of degree n, $f_0=1$, $f=2\lambda x$ and

$$(1-x^2)((f_n')^2-f_{n-1}'f_{n+1}')=n(n+2\lambda)f_n^2-(n+1)(n+2\lambda-1)f_{n-1}f_{n+1}$$

for each $n\geqq 1$, where $\lambda\neq-(n-1)/2$, then $f_n(x)=P_n^{(\lambda)}(x)$. This generalizes a result of the reviewer [Bull. Calcutta Math. Soc. **46** (1954), 93–95; MR **16**, 694]. Theorem 4. If $f_0=1$, $f_1=2x$ and

$$(f_n')^2=f_nf_n''=2n[f_n^2-f_{n-1}f_{n+1}]$$

for each $n\geqq 1$, then $f_n(x)=H_n(x)$.

L. Carlitz.

Ferrer Figueras, Lorenzo. On the recurrence relations deduced from the generating function

$$u_{3,3}=[1-x^3+(x-z)^3]^{-1/3}.$$

Gac. Mat., Madrid (1) **8** (1956), 252–255. (Spanish)

★**Scherwatow, W. G. Hyperbelfunktionen. Kleine Ergänzungsreihe zu den Hochschulbüchern für Mathematik. XVIII.** VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. iv+53 pp.

A translation of the book published by Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1954.

Myrberg, P. J. Über automorphe Thetafunktionen zweiter Ordnung bei fuchsschen Gruppen beliebigen Geschlechtes. Ann. Acad. Sci. Fenn. Ser. A. I. no. **225** (1956), 11 pp.

Let Γ be a Fuchsian group of analytic transformations on the unit disc $|z|<1$, containing no parabolic transformations. An automorphic theta function of second order for Γ [compare with the automorphic theta functions of first order considered by the author in, e.g., same Ann. no. **200** (1955); MR **17**, 603] is a function $g(z)$ holomorphic in $|z|<1$ and satisfying a functional equation of the form $g(Sz)=g(z)\exp[a_Su_1(z)+b_Su_2(z)+c_S]$ for any S in Γ; here a_S, b_S, c_S are constants, and $u_i(z)$ are holomorphic functions which in turn satisfy functional equations of the form $u_i(Sz)=\sum_j\alpha_{ij}(S)u_j(z)$, where $\alpha_{ij}(S)$ are constants forming a 2×2 matrix of determinant 1. The author determines all such functions, generalizing the results he had previously obtained for Fuchsian groups of genus zero [same Ann. Ser. A.I. no. **223** (1956), 11 pp.; MR **18**, 571].

R. C. Gunning (Princeton, N.J.).

Yu, Yi-Yuan. On the generalized ber, bei, ker, and kei functions with application to plate problems. Quart. J. Mech. Appl. Math. **10** (1957), 254–256.

The author points out that the solution of

$$(*)\quad \nabla^2\nabla^2y-s\nabla^2y+ty=0;\ \nabla^2\equiv\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}$$

when $s^2-4t<0$, may be expressed in terms of Bessel functions of complex argument and states that "since these reduce to the ber, bei, ker, and kei functions when s is zero, they are called the generalized ber, bei, ker, and kei functions." {Note: Author's results (including formulas for derivatives of the functions) may also be found elsewhere [e.g., Frederick, J. Appl. Mech. **23** (1956), 195–200]. Furthermore the fact that solutions of differential equations of the type (*) may be expressed in terms of Bessel functions of complex argument seems to have been known [e.g., author's own reference (1), footnote 7, p. 122].}

P. M. Naghdi (Ann Arbor, Mich.).

Thorne, R. C. The asymptotic expansion of Legendre functions of large degree and order. Philos. Trans. Roy. Soc. London. Ser. A. **249** (1957), 597–620.

Il s'agit de donner des développements asymptotiques pour les fonctions associées de Legendre $P_n^{-m}(z)$, $Q_n^{-m}(z)$ pour m et n grands $(0<m<n)$, le rapport $\alpha=m/(n+\frac{1}{2})$ étant supposé demeurer fixe $(0<\alpha<1)$, (z est une variable complexe quelconque). L'auteur obtient 3 catégories de développements. Dans le premier les éléments sont des fonctions exponentielles (leurs expressions sont d'ailleurs compliquées); on peut obtenir des développements analogues pour les dérivées. Ces développements sont valables pour $\Re(z)>0$ sauf si $|\Im(z)|<\varepsilon$, $\Re(z)<\sqrt{(1-\alpha^2)}+\varepsilon$

($\varepsilon>0$). Dans le second les éléments sont des fonctions de Airy obtenues à partir d'un transformation de l'équation de Legendre en l'équation $d^2W/d\zeta^2=[u^2\zeta+f(\zeta)]W$. Ce développement est valable pour tout le plan muni de la coupure $(+1, -\infty)$ sauf dans un domaine entourant le point $z=-1$ et dans la bande $|\Re(z)|<\beta+\varepsilon$, $0\geqq\Im(z)>-\varepsilon$. Le 3ème groupe de développements a pour éléments des fonctions de Bessel. On l'obtient comme application directe des résultats de l'étude précédente. MR 18, 477.

R. Campbell (Caen).

Mohr, Ernst. Nachtrag zu meiner Note "Die Maxwellsche Erzeugung der Kugelfunktionen". Math. Nachr. 15 (1956), 122.

An addition to the bibliography of the article reviewed in MR 16, 1021.

Kaufman, H. A generalization of the sine function. Amer. Math. Monthly 64 (1957), 181–183.

Let $I(x)$ be a continuous function defined on an interval (a, b). The generalization in question is given by defining functions $C_j(x, \xi)$ $(j=1, 2, \cdots, n)$ such that if $y(x)$ is a solution of the differential equation $d^ny/dx^n=I(x)y$, then $y(x+\xi)=\sum_{j=1}^n C_jy^{(j-1)}(x)$.

Deaux, R.; et Delcourte, M. Calcul des intégrales $(m, n)=\int_0^\infty x^{-n} \sin^m x dx$, m et n entiers positifs, $m\geqq n$. Mathesis 66 (1957), 16–22.

In the notation of the title, the function $2^{m-1}(n-1)!(m,n)$ is an integral multiple of $\frac{1}{2}\pi$ or the logarithm of a rational number according as m and n are of the same parity or not. In this note these integers and rationals are given explicitly as sums of $[\frac{1}{2}(m-1)]$ terms or products of this many factors. The method is to expand $(\sin x)^m$ in a Fourier series of cosines or sines according as m is even or odd and to integrate termwise by parts. *D. H. Lehmer.*

See also: Eichler, p. 17; Jankowski, p. 22; Carlitz, p. 29; Stone, p. 41; Rooney, p. 45.

Sequences, Series, Summability

Carlitz, L. The expansion of certain products. Proc. Amer. Math. Soc. 7 (1956), 558–564.

Let $G_m(x, y)$, $H_m(x, y)$, $K_m(x, y)$ be defined by

$$\prod_{m,n=0}^{\infty}(1+x^my^nt)=\sum_{m=0}^{\infty}\frac{t^mG_m(x, y)}{(x)_m(y)_m},$$

$$\prod_{m,n=0}^{\infty}\frac{1}{(1-x^my^nt)}=\sum_{m=0}^{\infty}\frac{t^mH_m(x, y)}{(x)_m(y)_m},$$

$$\prod_{m,n=0}^{\infty}\frac{(1+x^my^nt)}{(1-x^my^nt)}=\sum_{m=0}^{\infty}\frac{t^mK_m(x, y)}{(x)_m(y)_m},$$

where $(x)_m=\prod_{r=1}^m(1-x^r)$. Recursion formulas for G_m, H_m, K_m are developed, and it is shown that they are polynomials with integral coefficients, of degree $m(m-1)/2$ in each of x and y. Functional equations are also derived; for example, $H_m(x, y)=y^{m(m-1)/2}G_m(x, y^{-1})$. Special results are obtained for particular values of the arguments. {Reviewer's note: The expression for K_4 at the end of § 3 is incorrect, and so are the remarks which follow, concerning the negativity of some coefficients of G_4 and H_4.}

N. J. Fine (Philadelphia, Pa.).

Petersen, G. M. The iteration of regular matrix methods of summation. Math. Scand. 4 (1956), 276–280.

Regular matrix methods are b-equivalent if they sum the same bounded sequences; A is b-stronger than B if every bounded sequence summed by B is also summed by A; they are then summed to the same value. It is shown here that b-equivalent matrices can have quite different behaviors when iterated with other matrices. (For $A=(a_{mn})$, $B=(b_{mn})$ the iteration BA takes $\{s_n\}$ into $\{\sigma_n\}$ where $\sigma_k=\sum_{m=1}^\infty b_{km}t_m$, $t_m=\sum_{m=1}^\infty a_{mn}s_n$.) Thus: a) For A, B regular there is a C which is b-equivalent to A such that BC is b-equivalent to A. b) For any A and B, such that for some $\{n_k\}$, all bounded sequences vanishing off $\{n_k\}$ are summed to 0 by B, then for every bounded $\{s_n\}$ there is a C which is b-equivalent to A for which BC sums $\{s_n\}$. c) With A, B as above, if there is a C which is b-stronger than both A and B then there is an A' which is b-equivalent to A such that BA' is b-stronger than both A and B. *C. Goffman* (Norman, Okla.).

Petersen, G. M. The norm of iterations of regular matrices. Proc. Cambridge Philos. Soc. 53 (1957), 286–289.

For a regular matrix A, by $\|A\|$ we mean the number

$$\inf[\lim\sup_m \sum_{n=1}^{\infty}|b_{mn}|],$$

where $B=(b_{mn})$ varies over all matrices b-equivalent to A. This paper proves the interesting fact that if $A=(a_{mn})$ is regular and $\lim\sup_m \sum_{n=1}^\infty |a_{mn}|=M>1$ and if, for some $\{n_k\}$, $\lim \sum_{k=1}^\infty |a_{mn_k}|=0$, then there is a B which is b-equivalent to A such that $\|B^k\|\geqq M^{k-1}$, for every k. {Remark. A consequence is that A and B can be b-equivalent without the same holding for A^2 and B^2}.

C. Goffman (Norman, Okla.).

Petersen, G. M. Sets and sub-series. Canad. J. Math. 9 (1957), 223–224.

Zu jeder Reihe $\sum a_n$ mit $a_n\geqq0$ gibt es einen 'Index p' $(-\infty\leqq p\leqq+\infty)$, so daß $\sum a_n^q$ für $q>p$ konvergiert und für $q<p$ divergiert; dabei sollen die Fälle $p=\pm\infty$ sinngemäß interpretiert sein. Jeder durch einen nichtabbrechenden Dualbruch dargestellten Zahl $\xi=0, b_1b_2\cdots b_n\cdots$ aus (0, 1) kann eineindeutig die 'Teilreihe' $\sum \alpha_n$ von $\sum a_n$ zugeordnet werden, bei der $\alpha_n=0$ wenn $b_n=0$ und $\alpha_n=a_n$ wenn $b_n=1$ ist. Verfasser fragt nach der Häufigkeit derjenigen Teilreihen einer gegebenen Reihe $\sum a_n$, die einen verschiedenen Index haben. Satz: Ist $0<a_n\searrow$ und p der Index von $\sum a_n$, so ist die Menge der Punkte ξ aus (0, 1), die einer Teilreihe mit Index $\neq p$, dass heißt $<p$, entsprechen, von erster Kategorie. *D. Gaier.*

★Cassina, Ugo. Sulla formola sommatoria di Euler col resto di Malmsten. Scritti matematici in onore di Filippo Sibirani, pp. 49–61. Cesare Zuffi, Bologna, 1957.

Nella presente Nota mi propongo di trovare per una nuova via la formola sommatoria di Euler con il resto di Malmsten, che è quello che più si presta alle applicazioni numeriche. *Dall'introduzione.*

See also: Korenblyum, p. 46; Mettler und Weidenhammer. p. 82; Vasilache, p. 101.

Approximations, Orthogonal Functions

Berman, D. L. Convergence of Lagrange's interpolation process constructed for absolutely continuous functions and functions of bounded variation. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 9–12. (Russian)

Let $L_n(x)=\sum_{k=0}^n f(x_k^{(n)})l_k^{(n)}(x)$ be a sequence of Lagrange interpolation polynomials for a function $f(x)$ in $[-1,+1]$. Let (A) $|l_k^{(n)}(x)|\leq|l_{k+1}^{(n)}(x)|$ if $x_k^{(n)}\leq x_{k+1}^{(n)}\leq x$ and $|l_k^{(n)}(x)|\geq|l_{k+1}^{(n)}(x)|$ if $x\leq x_k^{(n)}\leq x_{k+1}^{(n)}$, and let (B) $|l_k^{(n)}(x)|\leq\varphi(h)$ for all k, n, where h is the number of interpolation points $x_i^{(n)}$ between x and $x_k^{(n)}$, and $\varphi(h)$ is a fixed decreasing function which tends to zero for $h\to\infty$. Then $L_n(x)\to f(x)$ at each point x of continnity of a function of bounded variation $f(x)$; the convergence is uniform for absolutely continuous f. (A), (B) hold in particular if the $x_k^{(n)}$ are zeros of Jacobi polynomials $J_n^{(\alpha_n,\beta_n)}(x)$ of orders $-1\leq\alpha_n$, $\beta_n\leq\lambda<0$ [Berman, same Dokl. (N.S.) **60** (1948), 333–336; MR **9**, 584]. *G. G. Lorentz*

Zuhovickiĭ, S. I. On approximation of real functions in the sense of P. L. Čebyšev. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 2(68), 125–159. (Russian)

Let Q be a compact metric space, let $C(Q)$ be the Banach space of real-valued continuous functions on Q with the uniform norm $\|\cdot\|_u$. Let $\varphi_1, \cdots, \varphi_n$ be given linearly independent functions in $C(Q)$, and let Φ be an arbitrary function in $C(Q)$. For every n-tuple $x=(\xi_1, \cdots, \xi_n)$ of real numbers, one defines

$$L(x)=\|\sum_{i=1}^n \xi_i\varphi_i-\Phi\|_u.$$

The Čebyšev approximation problem is the problem of minimizing $L(x)$. The present paper is a clearly organized survey of the presently known results concerning this problem and its ramifications.

The fundamental theorem used is the following. Let E be any real normed linear space, and let $x_1, \cdots, x_n$ be linearly independent elements of E. For every sequence $c_1, \cdots, c_n$ of real numbers not all zero, there is a bounded linear functional f on E such that $f(x_i)=c_i$ $(i=1, 2, \cdots, n)$ and $\|f\|$ is a minimum. This minimum is equal to

$$\max\left\{\frac{|\sum_{i=1}^n c_i\xi_i|}{\|\sum_{i=1}^n \xi_i x_i\|}: \text{all real } \xi_1, \cdots, \xi_n\neq 0, 0, \cdots, 0\right\}.$$

This theorem is applied to prove the existence of a minimizing $x^{(0)}$ for the function $L(x)$.

If $\phi_1, \cdots, \phi_n$, and Φ are linearly independent, then more can be said. There is a finite subset $\{q_1, \cdots, q_r\}$ of Q $(r\leq n+1)$ such that $\min\{L(x)$: all $x=(\xi_1, \cdots, \xi_n)\}$ equals $\min\{\max[|\Phi(q_j)-\sum_1^n \xi_i\phi_i(q_i)|: j=1, \cdots, r]$: all $x=(\xi_1, \cdots, \xi_n)\}$. Furthermore, one of the functions minimizing $L(x)$ on $\{q_1, \cdots, q_r\}$ also minimizes $L(x)$ on Q. This theorem in its general form is due to E. Ya. Remez.

The theorem of A. Haar [Math. Ann. **78** (1918), 294–311] concerning the uniqueness of the minimizing $(\xi_1, \cdots, \xi_n)$ is reproved and generalized to the present situation.

A number of related results are also given. Other references for the subject-matter of this survey are M. G. Kreĭn [Uspehi Mat. Nauk (N.S.) **6** (1951), no. 4(44), 3–120; MR **13**, 445]; E. Ya. Remez [Ukrain. Mat. Ž. **5** (1953), 3–49; MR **15**, 407]; N. I. Ahiezer [Lectures on the theory of approximation, OGIZ, Moscow-Leningrad, 1947; MR **10**, 33; **15**, 867]. *E. Hewitt.*

Zuhovickiĭ, S. I.; and Stečkin, S. B. On the approximation of abstract functions with values in a Banach space. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 773–776. (Russian)

Let X be a real Banach space and Q a compact metric space containing at least 2 points. Let $C(X)$ be the space of all strongly continuous mappings of Q into X (note the difference between this notation and the notation commonly used in the United States). For $\phi\in C(X)$, let $\|\phi\|=\max\{\|\phi(q)\|_X: q\in Q\}$. With this norm and with pointwise linear operations, $C(X)$ is clearly a real Banach space. Let $f_1, \cdots, f_N$ be linearly independent elements of $C(X)$ .Write $p=p_\alpha=\sum_{k=1}^N \alpha_k f_k$ $(\alpha_1, \cdots, \alpha_N$ real numbers). For $\phi\in C(X)$, let $\mathfrak{E}(\phi)=\inf_\alpha\|\phi-p_\alpha\|$. The problem considered is that of finding a $p=p^*$ such that $\|\phi-p^*\|=\mathfrak{E}(\phi)$. Theorem 1. Such a p^* always exists.

A point $q_0\in Q$ is said to be a point of maximal deviation for a polynomial p as above if $\|\phi(q_0)-p(q_0)\|_X=\|\phi-p\|$. Obviously every polynomial p admits at least one such point. Consider the following properties. (K_m) For every $\phi\in C(X)$ and every p^* as in Theorem 1, there are at least m points of maximal deviation. (P_m) For every m distinct points q_j of Q and any m elements x_j of X, there is a polynomial p such that $p(q_j)=x_j$ $(j=1, 2, \cdots, m)$. Theorem 2. A system $f_1, f_2, \cdots, f_N$ has property (K_m) if and only if it has property (P_{m-1}). Several other theorems, dealing with uniqueness of p^*, are also stated. No proofs are given. All results are said to hold if X is a complex Banach space. The present note supersedes earlier work by the authors and M. G. Kreĭn [Zuhovickiĭ and Stečkin, same Dokl. (N.S.) **106** (1956), 385–388; MR **18**, 222; Zuhovickiĭ, Mat. Sb. N.S. **37(79)** (1955), 3–20; MR **17**, 388; Zuhovickiĭ and Kreĭn, Uspehi Mat. Nauk (N.S.) **5**, (1950), no. 1(35), 217–229: MR **11**, 662]. *E. Hewitt.*

Davis, Philip; and Fan, Ky. Complete sequences and approximations in normed linear spaces. Duke Math. J. **24** (1957), 183–192.

Let X be a normed linear space, X^* its dual. The authors introduce two strengthened forms of completeness and prove several theorems involving them. (1) Given a sequence $\{a_n\}$ of nonnegative numbers, $\{f_n\}\subset X$ is called $\{a_n\}$-complete if $\phi=0$ is the only $\phi\in X^*$ satisfying $|\phi(f_n)|\leq a_n$ for every n. (2) Let $p\geq 1$; then $\{f_n\}$ is called complete of order p if $\{\sum|\phi(f_n)|^p\}^{1/p}<\infty$ implies $\phi=0$. Complete sequences of both kinds are characterized by approximation properties. An $\{a_n\}$-complete sequence $\{f_n\}$ can be constructed in a Banach space from a complete sequence $\{g_n\}$ such that $\limsup\|g_n\|^{1/n}=\sigma<\infty$ by taking $f_n=\sum_{k=1}^\infty z_n^k g_k$, where $0<|z_n|<1/\sigma$, $\lim z_n=0$, provided that $a_n^{1/n}=O(|z_n|)$. The authors prove several theorems of Paley-Wiener type, in which the completeness of a sequence implies the completeness of a "near-by" system. They give a number of examples. *R. P. Boas, Jr.*

Trigonometric Series and Integrals

Mordell, L. J. On Ingham's trigonometric inequality. Illinois J. Math. **1** (1957), 214–216.

The inequality is: If

$$f(t)=\sum_{n=-N}^{N'} a_n e^{-\lambda_n t i}$$

where the λ_n are real and $\lambda_n-\lambda_{n-1}\geq\gamma>0$ $(N<n\leq N')$, and

if $\gamma T=\pi$, then

$$|a_n|\leq\frac{K}{T}\int_{-T}^{T}|f(t)|dt \quad (N\leq n\leq N')$$

with a suitable constant K. In the special case $\lambda_n-\lambda_{n-1}=\gamma$ $(N<n\leq N')$ this is obviously true with $K=\frac{1}{2}$. In the general case the reviewer proved that it is true with $K=1$, and that this is the best possible absolute constant [Proc. Cambridge Philos. Soc. 46 (1950), 535–537; MR 12, 255, 1002]. The author of this note rearranges the proof with a somewhat more advantageous choice of an auxiliary function, and replaces K by a K_n depending (explicitly) on the differences $\lambda_m-\lambda_n$ $(N\leq m\leq N')$, where $K_n=\frac{1}{2}$ in the special case and $K_n<1$ in the general case. This brings all cases under one statement, and shows that the general inequality can be improved if K is not required to be an absolute constant. *A. E. Ingham.*

Csibi, S. Notes on de la Vallée Poussin's approximation theorem. Acta Math. Acad. Sci. Hungar. 7 (1956), 435–439. (Russian summary)

Let $f(x)$ be a continuous periodic function in $[a, b]$ and let E_n denote its order of approximation by trigonometric polynomials. If $E_n\leq n^{-p}\Omega(n)$, where p is a nonnegative integer, $\Omega(u)\geq 0$ decreases and is such that $\int^\infty u^{-1}\Omega(u)du<+\infty$, then according to de la Vallée-Poussin [Leçons sur l'approximation $\cdots$, Gauthier-Villars, Paris, 1919], $f(x)$ has a pth derivative with the modulus of continuity of the order

$$(*)\qquad \omega_p(\delta)=O\left(\delta\int_A^{A\delta^{-1}}\Omega(u)du+\int_{\delta^{-1}}^\infty u^{-1}\Omega(u)du\right).$$

In case of approximation by algebraic polynomials, (*) holds only in any closed subinterval of (a, b). Using Markov's inequality for the derivative of a polynomial, the author proves that (*) holds in the whole of (a, b) if $f(x)$ has the order of approximation by algebraic polynomials $\bar{E}_n\leq n^{-2p}\Omega(n^2)$. This result cannot be improved.

G. G. Lorentz (Detroit, Mich.).

Stečkin, S. B. On Fourier coefficients of continuous functions. Izv. Akad. Nauk SSSR. Ser. Mat. 21 (1957), 93–116. (Russian)

The author investigates conditions on a system of nonnegative functions $\{\Phi_n(u)\}$ which allow the existence of a continuous function $f(x)\sim\sum\rho_n\cos(nx-\alpha_n)$ with $\sum\Phi_n(\rho_n)=\infty$, or more generally with $\sum\Phi_k(\rho_{n_k})=\infty$. He proves the following general theorem which implies all previous results of this character. Let $\{n_k\}$ be an increasing sequence of integers and let $\{\Phi_k(u)\}$ be such that for some p, $0<p<2$, and for every k, either $\Phi_k(u)\geq 0, \Phi_k(u)\uparrow$, $u^{-p}\Phi_k(u)\downarrow$, or $\Phi_k(u)\geq 0$, $u^{-p}\Phi_k(u)\uparrow$, $u^{-2}\Phi_k(u)\downarrow$, $u^{-2}\Phi_k(u)\to\infty$ $(u\to 0)$ A necessary and sufficient condition for the existence of a power series $F(z)=\sum c_nz^n$, continuous in $|z|\leq 1$, with $\sum\Phi_k(|c_{n_k}|)=\infty$ is that for every positive ξ the series $\sum u_n^2(\xi)$ diverges, where $u_n(\xi)$ is the greatest root of $\Phi_n(u_n(\xi))=\xi u_n^2(\xi)$. Further consequences: (1) If $0<\varepsilon_k<2$ and $\{n_k\}$ is given, there is an F with $\sum|c_{n_k}|^{2-\varepsilon_k}=\infty$ if and only if $\sum\eta^{1/\varepsilon_k}$ diverges for every positive ε. (2) If also $\varepsilon_k\downarrow 0$ the last condition can be replaced by

$$\limsup\varepsilon_k\log k=\infty.$$

(3) If $d_n\geq 0$ and $0<\varepsilon_n<2$, there is an F such that $\sum d_n^{\varepsilon_n}|c_n|^{2-\varepsilon_n}$ diverges if and only if $\sum d_n^2\eta^{1/\varepsilon_n}$ diverges for every positive η; if $\varepsilon_n\geq\varepsilon>0$, the last condition is equivalent to $\sum d_n^2=\infty$. The following lemma on series has independent interest: let $\Phi_n(u)\geq 0$, $\Phi_n(u)\uparrow$, $u^{-2}\Phi_n(u)\downarrow$; then the divergence of the series $\sum u_n^2(\xi)$ described above is necessary and sufficient for the existence of a sequence $\{r_n\}$ of nonnegative numbers for which $\sum r_n^2<\infty$ and $\sum\Phi_n(r_n)=\infty$. *R. P. Boas, Jr.* (Evanston, Ill.).

Magarik, V. A. On summability $|C, \alpha|$ of Fourier series. Moskov. Gos. Univ. Uč. Zap. 181. Mat. 8 (1956), 183–196. (Russian)

The theorem proved in this paper can be stated as follows: The absolute summability $|C, \alpha|$ of a Fourier series by Cesàro means of any positive order $\alpha>0$ at a point $x=\xi$ is a local phenomenon. This theorem generalizes the corresponding result (Wiener) for $\alpha=0$, that is, for absolute convergence. *E. Kogbetliantz.*

Temko, K. V. Convex capacity and Fourier series. Dokl. Akad. Nauk SSSR (N.S.) 110 (1956), 943–944. (Russian)

Let $\lambda_0, \lambda_1, \lambda_2, \cdots$ be a convex sequence monotonically decreasing to 0 and such that $\sum\lambda_n=+\infty$, and let

$$Q(r, x, \lambda)=\tfrac{1}{2}\lambda_0+\sum_{n=1}^\infty r^n\lambda_n\cos nx \quad (0\leq r<1).$$

The author says that a Borel set E situated on $(0, 2\pi)$ has a positive convex capacity generated by $\{\lambda_n\}$ if there is a measure μ concentrated on E, with $\mu(E)=1$ and such that the integral

$$\int_E Q(r, x-y, \lambda)d\mu(y)$$

is bounded for $0\leq x\leq 2\pi$, $r<1$. When $\lambda_n=n^{-\alpha}$, $0<\alpha<1$, this coincides with the usual definition of α capacity, and for $\alpha=0$ we obtain logarithmic capacity. If $\{\lambda_n\}$ has the indicated properties, if $n\Delta\lambda_n$ does not increase and $\sum(a_n^2+b_n^2)(n\Delta\lambda_n)^{-1}$ converges, then the series

$$\sum(a_n\cos nx+b_n\sin nx)$$

can only diverge in a set of zero capacity generated by $\{\lambda_n\}$. *A. Zygmund* (Chicago, Ill.).

Kahane, Jean-Pierre; et Salem, Raphaël. Sur les ensembles de Carlsson et de Helson. C. R. Acad. Sci. Paris 243 (1956), 1706–1708.

The authors show that every symmetric perfect set E in $(0, 2\pi)$ carries complex-valued measures μ for which the ratio $\sup_n|\int_0^{2\pi}e^{nix}d\mu(x)|/\int_0^{2\pi}|d\mu(x)|$ is arbitrarily close to zero. The proof gives a method of constructing a continuous function on E which is not the restriction to E of an absolutely convergent Fourier series.

If a closed set E on $(0, 2\pi)$ has the property that every continuous function on E is the restriction to E of an absolutely convergent Fourier series, then E has in common with any arithmetic progression of N terms at most $[A\log N]$ points, where A is a constant depending only on E. An example shows that the logarithm cannot be replaced by a function of smaller growth. *H. Helson.*

Čereĭskaya, V. I. On uniform convergence of trigonometric series. Moskov. Gos. Univ. Uč. Zap. 181. Mat. 8 (1956), 159–163. (Russian)

L'A. retrouve, indépendamment et par une méthode plus directe, un test plus général que celui de Dini-Lipschitz, antérieurement établi par Salem [Nederl. Akad. Wetensch. Proc. Ser. A. 57 (1954), 550–555; MR 17, 845].

J. P. Kahane (Montpellier).

Doss, Raouf. On mean motion. Amer. J. Math. **79** (1957), 389–396.

For a trigonometric polynomial $p(t)=\sum_1^N a_n \exp(i\lambda_n t)$ we denote by $\arg^+ p(t)$ any branch of $\arg p(t)$ which is continuous, except at the zeros of $p(t)$, and which at a zero of order p is discontinuous with a jump $p\pi$ and has a value equal to the mean value of its limits from both sides. It is proved that the function

$$a^+(t)=\arg^+ p(t+1)-\arg^+ p(t)$$

is R.S.a.p., i.e. almost periodic in the sense of Riemann-Stepanoff, a notion previously introduced by the author [Compositio Math. 12 (1956), 271–283; MR 17, 1062]. As a consequence of this theorem it is proved that

$$\arg^+ p(t)=ct+\int_0^t \alpha(u)\,du+\psi(t),$$

where c is a constant, $\alpha(t)$ a Bohr a.p. function with mean value zero and $\psi(t)$ an R.S.a.p. function. *E. Følner.*

See also: Freud und Kra'lik, p. 26; Hartman, p. 46.

Integral Transforms

★ **Конторович, М.И. [Kontorovič, M. I.] Операционное исчисление и нестационарные явления в электрических цепях. [Operational calculus and non-stationary phenomena in electric circuits.]** Second edition, augmented. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 227 pp. 4.50 rubles.

This book contains a greatly abbreviated treatment of the application of the Laplace transformation to problems in electric circuit theory. For the most part, it is written for those who have completed only the calculus, but a few other topics (e.g. Fourier series and integrals, and contour integration) would be necessary at times. Most of the book is devoted to a strictly formal, almost cursory, introduction to the Laplace transformation, with many examples showing its application to simple electric circuits. Problems involving long lines (cables), and four terminal networks are given somewhat more extensive treatments. The Mellin inversion integral is introduced formally, but used very little. More attention is given to the Fourier transform and its applications. {In spite of the limitation imposed by the title of the book, it seems unfortunate that a few simple but useful extensions to mechanical and electro-mechanical systems were not made.} *R. E. Gaskell* (Seattle, Wash.).

Ditkin, V. A. Operational calculi for functions defined on the entire straight line. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 191–194. (Russian)

Let S be the set of functions of a real variable integrable over every bounded interval and having both left and right-hand Laplace transforms. Let S_+, S_- be the subsets of S consisting of functions zero for sufficiently small and large numbers respectively, and let M be the linear space of pairs $(\bar f_1, \bar f_2)$ where $\bar f_1$, $\bar f_2$ are Laplace transforms of functions in S_+, S_- respectively. (Here $\bar f_1$ and $\bar f_2$ are functions of a complex variable p.) Let M_0 be the subset of M consisting of pairs $(\bar f, \bar f)$ where $\bar f$ is a Laplace transform over a finite interval, and let $\bar S$ be M/M_0. Then the author proves that there is a one-one mapping of S onto $\bar S$.

A pair $(F_1(p), F_2(p))$ defines an operator F in M with domain $\bar\Omega_F$ consisting of those pairs $(\bar f_1, \bar f_2)$ for which $(F_1\bar f_1, F_2\bar f_2)$ is in M. Let Ω_F be the image of $\bar\Omega_F$ in S. Then, if for f in Ω_F we write $Ff=g$, where the image of g in $\bar S$ is the result of applying F to the image of f, an operator, again called F, is defined on Ω_F, which, however, is multivalued since it may take the null function into any element of S whose image in $\bar S$ is equal to the result of applying F to an element of M_0. *J. L. B. Cooper.*

★ **Janet, Maurice. Compléments divers sur la transformation de Laplace et les équations aux dérivées partielles.** Cours complémentaire (mars-avril-mai 1952). 2e éd. corrigée. Secrétariat Mathématique, 11 rue Pierre Curie, Paris, 1957. 47 pp. (polycopiées)

Supplements to the work reviewed in MR 17, 480.

Pollard, Harry. Representation as a Poisson transform. Trans. Amer. Math. Soc. **85** (1957), 174–180.

The author gives necessary and sufficient conditions that a function $f(x)$, defined for all real values of x, admit representation as a Poisson transform

$$(1) \qquad f(x)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dA(t)}{1+(x-t)^2}\ (-\infty<x<+\infty),$$

where A is a nondecreasing function. We define

$$(1.1) \qquad \tilde f(z)=-\pi^{-1}\int_{0+}^{\infty} u^{-2}[f(z+u)-2f(z)+f(z-u)]du,$$

$$(1.2) \qquad (T_tf)(x)=\tfrac12[f(x+it)+f(x-it)]+\frac{1}{2i}\int_{x-it}^{x+it}\tilde f(u)du,$$

supposing $f(z)$ analytic in the strip $|y|<1$, and $-1<t<1$, $-\infty<x<\infty$.

The author proves the following theorem. Let $f(x)$ be defined for all real values of x. In order that it admit the representation (1) with A a nondecreasing function it is necessary and sufficient that: (i) $f(x)$ admit analytic continuation into the strip $|y|<1$ of the z-plane; (ii) the integral in (1.1) converge uniformly in compact subsets of $|y|<1$ to a function $\tilde f(z)$; (iii) $(T_tf)(x)\geqq 0$, $0\leqq t<1$, $-\infty<x<\infty$; and (iv) $f(x+iy)=o(x^2)$, $|x|\to\infty$, uniformly in the substrips $|y|\leqq\delta$ for each $\delta<1$. *W. Saxer.*

See also: Babenko, p. 20; Stein, p. 23; Kahane et Salem, p. 31; Rooney, p. 45; Rudin, p. 46.

Ordinary Differential Equations

Wintner, Aurel. On the local uniqueness of the initial value problem of the differential equation $d^nx/dt^n=f(t, x)$. Boll. Un. Mat. Ital. (3) **11** (1956), 496–498.

The following generalization of Nagumo's uniqueness theorem is proved: Let $R=\{(t, x): 0<t\leqq a,\ -b\leqq x\leqq b\}$, $R_0=R\cup(0, 0)$, $f(t, x)$ be a real continuous function in R_0 such that $|f(t, x')-f(t, x'')|\leqq n!|x'-x''|t^{-n}$, where (t, x'), $(t, x'')\in R$. To any set of real constants $c_1, \cdots, c_{n-1}$, there exists a positive t_0 such that $d^nx/dt^n=f(t, x)$ has at most one solution $x(t)$ in $0\leqq t\leqq t_0$ satisfying $x(0)=0$, $d^hx/dt^h=c_h$ $(h=1, \cdots, n-1)$. The theorem remains obviously true if x, f, c_h are vectors. *J. L. Massera* (Montevideo).

Gol'dberg, A. A. On one-valued integrals of differential equations of the first order. Ukrain. Mat. Ž. **8** (1956), 254–261. (Russian)

Aržanyh, I. S. A system of ordinary differential equations which permits a potential method of integration. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. 16 (1955), 28–33. (Russian)

Utz, W. R. A third order differential equation. Monatsh. Math. 60 (1956), 329–332.

Let $f(x)$ be a real differentiable function, $\lim_{x\to\infty} f(x)=c$ $(0<c<\infty)$, $f'(x)$ bounded as $x\to\infty$. Let $p(x)$ be a real differentiable function and $0<a<\infty$, $p(a)=0$, $p'(a)>0$. If $y(x)$ is a solution of $fy'''-yy''+p(y')=0$ for which $\lim_{x\to\infty} y'(x)=a$, then $y=ax+b$, where b is a constant.

J. L. Massera.

Dobrotin, D. A. An estimate of the solutions of certain non-linear differential equations in the domain of asymptotic stability. Prikl. Mat. Meh. 20 (1956), 723–732. (Russian)

The basic equation under investigation is the real equation

$$(1)\qquad x^{(n)}+a_1x^{(n-1)}+\cdots+a_nx=f(x,t),$$

where the a_k are real constants, the characteristic equation has all its roots with negative real parts, and for $|x|\leq L$:

$$|f(x,t)|\leq A|x|^k\ (k>1),\quad |f_t|<B|x|^\gamma\ (\gamma>0).$$

If $\lambda_k=-\lambda_k'+i\lambda_k''$ are the characteristic roots, let $\lambda=\inf\{\lambda_k'\}$, and take $\beta\geq\lambda/k$. Then the estimate under consideration with a certain limitation on the initial values is of the form $|x(t)|\leq Ne^{-\beta t}$. Let $z(t)$ be the general solution of the homogeneous equation $(f=0)$ and $z_0(t)$ the special $z(t)$ for which $z_0^{(i)}(0)=0$, $i<n-1$, $z_0^{(n-1)}(0)=1$. Then the solution $x(t)$ of (1) with the same initial values as $z(t)$ satisfies

$$(2)\qquad x(t)=z(t)+\int_0^t z_0(t-t')f(x,t')dt'.$$

The estimate is obtained by an arduous process of successive approximation operating on (2). *S. Lefschetz.*

Hellman, O. Ein Verfahren zur Bildung von Matrizanten. Z. Angew. Math. Mech. 37 (1957), 139–144. (English, French and Russian summaries)

Let the solution of the system $d\mathfrak{z}/dt=A(t)\mathfrak{z}$, $\mathfrak{z}(0)=\eta_0$, be written $\mathfrak{z}=\Omega_0^t(A)\eta_0$. Putting

$$A=A_\varepsilon=\begin{pmatrix} a & \varepsilon\varphi \\ \varepsilon\xi & b\end{pmatrix},$$

where a and b are square matrices, the author considers the expansion of Ω in powers of ε. The procedure is recommended for physical cases of weakly coupled vibrations. *F. V. Atkinson* (Canberra).

Segre, Beniamino. Sui sistemi di equazioni differenziali lineari a coefficienti costanti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 20 (1956), 271–277.

Let α_{ij} $(i=1,\cdots,n;\ j=1,\cdots,m)$ denote a linear ordinary differential operator, with constant coefficients, u_i's given functions. The main result of the paper is the following: The differential system

$$\sum_{j=1}^m \alpha_{ij}y_j=u_i\quad (i=1,\cdots,n)$$

admits some solution $(y_1,\cdots,y_m)$ — when the rank r of the matrix $\|\alpha_{ij}\|$ is less than n — if and only if the given functions u_i satisfy the differential condition

$$\sigma_1u_1+\cdots+\sigma_nu_n=0,$$

with respect to any set $\sigma_1,\cdots,\sigma_n$ of linear differential operators with constant coefficients, such that

$$\sum_{i=1}^n \alpha_{ij}\sigma_i=0\quad (j=1,\cdots,m).$$

This and other theorems of the paper, which complete previous results by Picone and Ghizzetti [same Rend. (8) 19 (1955), 195–199; MR 18, 38], are obtained by using elegant algebraic arguments based on the isomorphism between the commutative ring of linear differential operators with constant coefficients and the ring of their characteristic polynomials. *G. Fichera* (Rome).

Wintner, Aurel. On criteria for linear stability. J. Math. Mech. 6 (1957), 301–309.

Consider the differential system

$$(1)\qquad x'=A(t)x,$$

where $A(t)$ is a continuous function of t for large t. If $\lambda(t)$ and $\mu(t)$ denote the least and largest eigenvalue, respectively, of the matrix $A^0=\frac{1}{2}(A+A^T)$, then

$$(2)\qquad \lambda(t)\leq\frac{d}{dt}\log|x(t)|\leq\mu(t)$$

for every nontrivial solution $x(t)$ of (1). The inequality (2) was discovered previously by the author [Amer. J. Math. 68 (1946), 553–559; MR 8, 272] and the first part of the present paper discusses the consequences of (2) pertaining to the stability of the solutions of (1). The second part of the paper deals with the preservation of stability under small perturbations of the matrix $A(t)$. From a previous result of the author [ibid. 68 (1946), 185–213, pp. 199–201; MR 8, 71] and a method of proof due to L. Cesari [Ann. Scuola Norm. Sup. Pisa (2) 9 (1940), 163–186; MR 3, 41], he proves that if $T(t)$ is a continuous matrix function of t for large t, $T(\infty)=\lim_{t\to\infty}T(t)$ exists, $\det T(\infty)\neq 0$, $\int_{t_0}^\infty|dT(t)|<\infty$, and $\limsup|x(t)|<\infty$ as $t\to\infty$ for all $x(t)$ satisfying (1), then the solutions of the equations $z'=T^{-1}ATz$ are in asymptotic one-to-one correspondence with the solutions of (1). *J. K. Hale.*

Kurcvell', Yaroslav. On the inversion of the second theorem of Lyapunov on stability of motion. Czechoslovak Math. J. 6(81) (1956), 217–259, 455–484. (Russian)

Consider $\dot x=f(t,x)$, $f(t,0)\equiv 0$, with $x\in R^n$ and f of class C on $G\times L$, where G is an open set in R^n, $0\in G$, and $L=[0,\infty)$. On G define

$$\omega(x)=\max\{\|x\|,\ \rho^{-1}(x,F)-2\rho^{-1}(0,F)\},$$

where $F=R^n-G$ and $\rho(x,F)=\inf\{\|x-z\|\,|\,z\in G\}$; if $G=R^n$ then $\omega(x)\equiv\|x\|$. The trivial solution $x=0$ is called strongly stable if given any positive δ, ε there exist positive $\Delta(\delta)$, $T(\delta,\varepsilon)$ with $\Delta\to 0$ monotonically as $\delta\to 0$ such that a continuation $x(t)$ of a solution $y(t)$ on $[t_0,t_1]\subset L$ with $\omega(y(t_0))\leq\delta$ satisfies $\omega(x(t))<\Delta$ on $t\geq t_0$ and $\omega(x(t))<\varepsilon$ on $t\geq t_0+T$. The main result proved by the author is this converse of Lyapunov's second theorem [cf., e.g., Problème général de la stabilité du mouvement, Princeton, 1947, pp. 259–262; MR 9, 34]: If $x=0$ is strongly stable in G, there exists a positive definite function $V(t,x)$ on $G\times L$, of class C^∞ on $G\times(0,\infty)$, $V\to 0(\infty)$ uniformly on L as $\omega(x)\to 0(\infty)$, such that $\dot V[t,x(t)]$ is negative definite; if f is independent of t, so is V. Among the further results, too detailed to be stated here, is the following: If $x=0$ is strongly stable in G, there exists a topological mapping of

G on the unit sphere which is of class C^∞ and has non-zero Jacobian everywhere.

The methods of proof are entirely independent of those of recent results by Massera who raised the problem as to the weakest hypotheses on f under which a converse to Lyapunov's theorem can be proved and who solved it very satisfactorily for equations in spaces of finite dimension [Ann. of Math. (2) **64** (1956), 182–206; MR **18**, 42] as well as of infinite dimension [Rev. Un. Mat. Argentina **17** (1955), 135–147; MR **18**, 900].

H. A. Antosiewicz (Washington, D.C.).

Krasovskiĭ, N. N. On the theory of the second method of A. M. Lyapunov for the investigation of stability. Mat. Sb. N.S. **40(82)** (1956), 57–64. (Russian)

Consider the system $\dot{x}=X(x, t)$, x, X being n-vectors, $X(0, t)=0$; X is a function of class C^1 for $\|x\|<H$, $t\geqq 0$, the derivatives being uniformly bounded. Let $F(x_0, t_0, t)$ be the solution through (x_0, t_0). The behavior of the trajectories in the ε-neighborhood of 0, $\varepsilon<H$, will be said to be non-critical and uniform if for each $\eta>0$ there is a $T(\eta)$ such that for any (x_0, t_0), $\|x_0\|\geqq\eta$, $t_0\geqq T(\eta)$, there is a $t(t_0, x_0)$, $|t(t, x_0)|<T(\eta)$, such that $\|F[x_0, t_0, t_0+t(t_0, x_0)]\|\geqq\varepsilon$, in the case of linear equations with constant coefficients this is equivalent to saying that all characteristic roots have non-vanishing real parts; in the case of asymptotic stability the condition is equivalent to the uniformity of asymptotic stability with respect to (x_0, t_0). 1) A necessary and sufficient condition for the existence of a Lyapunov function having an infinitely small upper bound and a positive definite total derivative is that the behavior be non-critical and uniform. 2) A necessary and sufficient condition for the existence of a Lyapunov function satisfying the conditions of the first (Lyapunov) theorem on instability is that the behavior be noncritical and uniform and that there exist a $t_0>0$ and a sequence $x_n\to 0$ such that $t(t_0, x_n)>0$. 3) The non-critical uniform behavior is rough in the following sense: There exists a positive definite continuous function $\eta(x)$ such that the solutions of

$$\dot{x}=X(x, t)+R(x, t)$$

also have a non-critical uniform behavior provided that $\|R(x, t)\|\leqq\eta(x)$. *J. L. Massera* (Montevideo).

Zoobow, W. J. Conditions for asymptotic stability in case of non-stationary motion and estimate of the rate of decrease of the general solution. Vestnik Leningrad. Univ. **12** (1957), no. 1, 110–129, 208. (Russian. English summary)

The author first defines for

$$\dot{x}=X(x, t) \ (x \text{ an } n\text{-vector}),$$

where X is defined for all x, t and satisfies the standard existence and uniqueness conditions, various types of asymptotic stability (a.s.). Def. 1. The a.s. of the origin as usual. Def. 2. When the origin has a.s. then it has uniform a.s. if there exists an $L(z)$ $(z\in[-\infty, +\infty])$ which is strongly monotone decreasing on $[-\infty, 0]$, which $\to 0$ as $z\to+\infty$, also an $\varepsilon>0$, such that for $\|x_0\|<\varepsilon$ (Euclidean norm) $\|x(t, x_0, t_0)\|<L(t-t_0)$. Def. 3. If the origin has a.s. then it is uniformly attractive whenever for every $x_0\neq 0$ there are τ and $\alpha(x_0, \tau)$ such that any solution $x(t, x_0, t_0)$ satisfies $\|x\|>\alpha>0$ for every $t_0>0$, and every

$$t\in[t_0, t_0+\tau].$$

Def. 4. The region A of a.s. is the set of all (x_0, t_0) such that $x(t, x_0, t_0)\to 0$ as $t\to+\infty$. The set A consists of complete trajectories [Erugin, Prikl. Mat. Meh. **15** (1951), 227–236; MR **12**, 705]. It is generally assumed that A includes the whole "axis" $x=0$.

A number of rather complicated theorems interrelating these various concepts are given. Noteworthy and relatively simple is the theorem: In order that A be the region of a.s., of uniform a.s., and uniformly attractive, it is necessary and sufficient that there exist two functions $V(x, t)$ defined and continuous in A, $W(x)$ defined and continuous for all x, such that: 1. V is definite negative and

$$W(x)>\alpha_1(\beta)>0$$

for $\|x\|>\beta>0$; (2) $V\to 0$ as $x\to 0$ uniformly in t for $t\geqq 0$ and $W(0)=0$; (3) $\dot{V}(x(t, x_0, t_0)=W(x_0)$; (4) $V(x, t)\to-\infty$ when (x, t) tends to a finite point of $\bar{A}-A$.

Noteworthy necessary and sufficient conditions for a.s. of the origin are given in the paper. *S. Lefschetz*.

Frid, I. A. On the stability of solutions of a linear differential equation with retardation in the critical case. Moskov. Gos. Univ. Uč. Zap. **181.** Mat. **8** (1956), 73–82. (Russian)

The extension of the Poincaré-Lyapunov stability theory to nonlinear differential-difference equations has been considered by the reviewer [Ann. of Math. (2) **50** (1949), 347–355; MR **10**, 715] and by E. M. Wright [Proc. Roy. Soc. Edinburgh. Sect. A. **63** (1949–50), 18–26; MR **12**, 106]. The author considers the problem under the assumption that a finite number of roots of the characteristic equation have real part zero, and the remainder have real parts which are uniformly negative.

R. Bellman (Santa Monica, Calif.).

Tusov, A. P. On the stability in the large of a certain regulation system. Vestnik Leningrad. Univ. **12** (1957) no. 1, 57–75, 209. (Russian. English summary)

The system examined is of the form

$$\dot{x}_i=\sum a_{ik}x_k+f(x_2) \ (i, k=1, 2, 3),$$

where the a_{ik} are real and f satisfies the standard existence and unicity conditions in the whole space and furthermore $\alpha x_2^2<x_2f(x_2)<\beta x_2^2$, where α, β are the extreme values of the constant a, for which upon replacing f by ax_2, the characteristic roots all have negative real parts. The author gives conditions based upon the values of certain functions (determinants and the like) of the a_{ik}, under which the system is stable in the large. The main tool is the construction of a function of Liapunov by Lur'ye and Postnikov [Prikl. Mat. Meh. **8** (1944), 246–248] together with this proposition: Take the n-vector system $\dot{x}=X(x)$, where X satisfies the standard existence and unicity conditions in any finite domain and $X(0)=0$. Suppose that there exists a positive Liapunov function V such that: (a) there exist $M, N>0$ such that if $\sum x_s^2\geqq N$ then $V\geqq M$; (b) in $V<M$, $\dot{V}<0$. Then any motion starting in $V<M$ remains there.

The problem discussed is a special case of one proposed by Aizerman [Uspehi Mat. Nauk (N.S.) **4** (1949), no. 4(32), 187–188]. Analogous systems in two dimensions and special three-dimensional systems have been discussed by Erugin [Prikl. Mat. Meh. **14** (1950), 459–512, 659–664; **16** (1952), 620–628; **19** (1955), 599–616; MR **12**, 412; **14**, 376; **17**, 366], Malkin [ibid. **16** (1952), 365–368; MR **14**, 48], Krasovskii [ibid. **16** (1952), 547–554; **17** (1953), 339–350, 651–672; MR **14**, 376, 1087; **15**, 624], Tuzov [Vestnik Leningrad. Univ. **10** (1955), no. 2, 43–70; MR **16**, 1025]. *S. Lefschetz* (Mexico City).

LaSalle, J. P. A study of synchronous asymptotic stability. Ann. of Math. (2) **65** (1957), 571–581.

The author considers physical systems in which the applied forces and the parameters which characterize the system are periodic in time of period θ. Let R be the space of all possible states $P, Q, \cdots$, of the system. R is assumed to be a metric space with metric ρ. The changes in the state of the system are described by a function, $f(t, t_0, P)$, with $f(t_0, t_0, P)=P$. The motion or orbit $M(P)$ through P is the sequence

$$M(P)=[P, P^2=T(P), \cdots, P^n=T^{n-1}(P), \cdots],$$

where $T(P)=f(t_0+\theta, t_0, P)$. P is asymptotically equivalent to Q if and only if the motion $M(P)$ through P is asymptotic to the motion $M(Q)$ through Q, i.e., $\rho(P^n, Q^n)\to 0$ as $n\to\infty$. This equivalence relation defines a partition of the phase space R and it is the character of this partition which determines the asymptotic stability properties of the system. The interior points of the partition are the asymptotically stable points and the boundaries of the partition are the separatrices. This partition is then used to define the asymptotic stability of sets and a measure of their stability. These concepts are then applied to the classification and study of asymptotically stable motions. For example, it is shown that Trefftz's stability is the requirement that a limit point of the motion be asymptotically stable and this implies the limiting motion is periodic. This is a "strong" type of stability and the author considers other "weaker" types of stability.

J. K. Hale (St. Paul, Minn.).

Urabe, Minoru. Reduction of periodic system to autonomous one by means of one-parameter group of transformations. J. Sci. Hiroshima Univ. Ser. A. **20** (1956), 13–35.

Consider the periodic system of differential equations (1) $dx_i/dt=\sum_{j=1}^n c_{ij}(t)x_j+\sum_{(s)}'' c_{is}(t)x_1^{p_1}\cdots x_n^{p_n}$ $(i=1, \cdots, n)$ where $(s)=(p_1, \cdots, p_n)$ ranges over all values $p_j\geqq 0$, $\sum_{j=1}^n p_j\geqq 2$, and where the c's are continuous in t for $-\infty<t<\infty$ and periodic of period $\omega>0$; and also the autonomous system

$$(2)\quad dx_i/dt=\xi_i(x)=\sum_{j=1}^n c_{ij}x_j+\sum_{(s)}'' c_{is}x_1^{p_1}\cdots x_n^{p_n}\,(i=1, \cdots, n).$$

These two systems have many parallel properties, and the author proposes to explain this by showing how (in certain cases) (1) can be transformed into (2) by a transformation of type

$$(3)\qquad x_i=F_i(y, t)=\sum_{j=1}^n k_{ij}(t)y_j+\sum_{(s)}'' k_{is}(t)y_1^{p_1}\cdots y_n^{p_n},$$

with the k's continuous and with period ω, $\det|k_{ij}(t)|\neq 0$ (all t), and $F_i(y, 0)=y_i$.

Suppose (3) exists. Let $\varphi_i(x, t)[=\varphi_i(x_1, \cdots, x_n, t)]$ be the solution of (1) for initial conditions $\varphi_i(x, 0)=x_i$, $i=1, \cdots, n$, and define functions ψ_i by

$$(4)\qquad \psi_i(x, t)=G_i\{\varphi(x, t), t\},$$

where $y_i=G_i(x, t)$ is the inverse of (3). Then

$$\psi_i(x, \omega)=\varphi_i(x, \omega),$$

so that the transformation (5) $x_i'=\varphi_i(x, \omega)$ $(i=1, \cdots, n)$ is imbedded in the one-parameter group of transformations $x_i'=\psi_i(x, t)$ whose operator functions are the $\xi_i(x)$ of equations (2). [If the operator of a group is

$$\sum_{j=1}^n \xi_i(x)\frac{\partial}{\partial x_j}$$

then the $\xi_i(x)$ are the operator functions.] Moreover, (3) takes the form $x_i=\varphi_i\{\psi(y, -t), t\}$. Conversely, let the transformation (5), where φ_i has the same meaning, be imbedded in a one-parameter group $x_i^1=\psi_i(x, t)$ with operator functions $\xi_i(x)$. Use these $\xi_i(x)$ to form the autonomous system (2). Define the functions $F_i(y, t)$ by $F_i(y, t)=\varphi_i\{\psi(y, -t), t\}$. Then transformation (3) does carry (1) into (2). Thus, system (1) can be carried into (2) by a transformation of type (3) if and only if $x_i'=\varphi_i(x, \omega)$ can be imbedded in a one-parameter group of transformations whose operator functions are regular at the origin and vanish there. And (3) is then uniquely determined by the conditions $F_i(y, 0)=y_i$.

This result is used to obtain the known reduction of a linear periodic system to autonomous form. Then a closer study is made of the non-linear case (1) to determine when the above criterion is satisfied. For this purpose the author makes use of his earlier work [same J. **14** (1950), 115–126; **15** (1951), 25–37; MR **13**, 747]. A further application is made to the stoboscopic method which replaces the real periodic system $dx_i/dt=\varepsilon X_i(x, t, \varepsilon)$ by the approximating system $dx_i/dt=\varepsilon/\omega\int_0^\infty X_i(x, \tau, 0)d\tau$.

I. M. Sheffer (University Park, Pa.).

Hukuhara, Masuo. Sur les équations différentielles périodiques non linéaires. J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1957), 437–447.

Systems of first order non-linear periodic differential equations, $dx_j/dt=f_j(t, x_1, \cdots, x_n)$, f_j periodic in t, $j=1, 2, \cdots, n$, are considered. It is shown that suitable linear transformations may be found such that the resulting system may be formally integrated. It is then shown under what condition the formal solution is convergent. Further, discussion is given on the existence of periodic solutions.

Hirsh Cohen (Troy, N.Y.).

Saito, Tosiya. On center-type singular points. Kōdai Math. Sem. Rep. **7** (1955), 89–96.

Let $\dot{x}=y$, $\dot{y}=-x+P(x, y)$, where P is a polynomial. Several necessary and sufficient conditions which ensure that the origin is a center are derived for particular forms of P; some of them are already known, although the author does not mention this fact. (i) If

$$P=Ax^2+Bxy+Cy^2,$$

the condition is $B(A+C)=0$; (ii) if

$$P=(x^2+y^2)^N(Ax^2+Bxy+Cy^2)\ (N>1),$$

the condition is either $B=0$ or $A=C=0$; (iii) if

$$P=(x^2+y^2)^N(Ax^3+Bx^2y+Cxy^2+Dy^3)\ (N\neq 1),$$

the condition is $B=D=0$; (iv) if

$$P(\theta)=P(\cos\theta, \sin\theta)=A\cos(2s\theta)+B\sin(2s\theta)\ (s\geqq 1),$$

the condition is either $A=0$ or $B=0$ (except for some particular values of the degree n of $P(x, y)$); (v) if $P(\theta)=A\cos[(2s+1)\theta]+B\sin[(2s+1)\theta]$ $(s\geqq 1)$, the condition is $B=0$ (except for some particular values of n).

J. L. Massera (Montevideo).

Landis, E. M.; and Petrovskiĭ, I. G. On the number of limit cycles of the equation $dy/dx=P(x, y)/Q(x, y)$, where P and Q are polynomials of degree n. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 748–751. (Russian)

In two previous papers [same Dokl. **102** (1955), 293–2; Mat. Sb. N.S. **37(79)** (1955), 209–250; MR **16**, 1110; **17**, 364] the authors dealt with the number of limit-cycles of

a system $dy/dx=P(x,y)/Q(x,y)$, where P, Q are quadratic polynomials. This is now extended to the case where P, Q are polynomials of any degree n. They show that for n odd the number of limit-cycles is at most $\frac{1}{2}(9n^3+n^2-6n+6)$ while for n even it is at most $\frac{1}{2}(9n^3-4n^2-27n+14)$. The authors note that Otrokov [ibid. **34**(76) (1954), 127–144; MR **16**, 130] has given an example where for n odd there are $\frac{1}{2}(n^2-5n-26)$ limit-cycles, while for n even there are $\frac{1}{2}(n^2+5n-14)$.

S. Lefschetz (Mexico City).

Andronov, A. A.; and Leontovič, E. A. The generation of limit cycles from a fine (multiple) focus or center, or from a fine (multiple) limit cycle. Mat. Sb. N.S. **40**(82) (1956), 179–224. (Russian)

Consider the system (A) $\dot{x}=P(x,y)$, $\dot{y}=Q(x,y)$, $P(0,0)=Q(0,0)=0$, and the perturbed system ($\tilde{A}$) $\dot{x}=\tilde{P}(x,y)$, $\dot{y}=\tilde{Q}(x,y)$. Assume that $P, Q\in C^N$ (or are analytic), $N\geqq 2k+1$, and that $\tilde{P}, \tilde{Q}\in C^{2k+1}$, $\tilde{P}-P$ and $\tilde{Q}-Q$ and its derivatives up to the order $2k+1$ being $<\delta$, δ a sufficiently small positive number. Consider first the case where $(0,0)$ is a focus or center of (A) and set $\psi(\rho_0)=\rho_1-\rho_0$, where (ρ,φ) are polar coordinates and ρ_1 is the value of ρ corresponding to the argument 2π on the solution of (A) which starts at $(\rho_0, 0)$; the focus will be called multiple of order k if $\rho=0$ is a root of order $2k+1$ of ψ. In this case, there exist $\varepsilon_0>0$, $\delta_0>0$ such that if $\delta<\delta_0$ there are at most k cycles of ($\tilde{A}$) contained in the ε_0-neighborhood of $(0,0)$ and it is possible to construct systems such that the number of cycles is precisely k. An entirely similar situation takes place in the neighborhood of a multiple limit cycle, with the only difference that such a cycle will be called multiple of order r if 0 is a root of order r of the function $\psi(n)$ defined in a similar way as before, n being the minimum distance of any point to the limit cycle.

J. L. Massera (Montevideo).

★ **Gomory, Ralph E. Critical points at infinity and forced oscillation.** Contributions to the theory of nonlinear oscillations, vol. 3, pp. 85–126. Annals of Mathematics Studies, no. 36. Princeton University Press, Princeton, N. J., 1956. $4.00.

Let d/dt be denoted by $'$. The author considers the cases (a) $x'=X(x,y)$, $y'=Y(x,y)$; (b) $x''+f(x)x'+g(x)=0$; and (c) the case (b) with the periodic term $E(t)$ in place of 0. He dispenses with the reviewer's condition of "dissipative for large displacements" for the existence of periodic solutions by making a careful study of the critical point at infinity for (a). This method is then used to study a wide class of equations (b). Finally the procedure is applied to show the existence of periodic solutions for (c).

N. Levinson (Cambridge, Mass.).

Loud, W. S. Boundedness and convergence of solutions of $x''+cx'+g(x)=e(t)$. Duke Math. J. **24** (1957), 63–72.

The author gives bounds for $|x(t)|$ and $|x'(t)|$ for the equation $x''+f(x)x'+g(x)=e(t)$ when $f(x)\geqq c>0$, $g'(x)\geqq b$, $|e(t)|\leqq E$ and $g(0)=0$. For the case where $f(x)=c$, $e(t)$ is periodic and the condition $c^2>2H(A)$ is satisfied, where H is a bound for $|g'(x)|$ on an appropriate interval $|x|\leqq A$, he proves that all solutions in an appropriate (x,x') rectangle converge to one periodic solution as $t\to\infty$.

N. Levinson (Cambridge, Mass.).

Loud, W. S. Behavior of certain forced nonlinear systems of second order under large forcing. Duke Math. J. **24** (1957), 235–247.

Conditions are given which ensure that the equation

$$x''+(c+f(x))x'+kx+g(x)=Ae(t),$$

where $'\equiv d/dt$ and $e(t)$ has period L, should possess a unique solution of period L when A is sufficiently large. One set of conditions: (i) $f(x)$ and $F(x)=\int_0^x f(u)du$ bounded, $f(x)\to 0$ as $|x|\to\infty$; (ii) $(g(x)-g(y))/(x-y)$ is bounded and $\to 0$ as $x,y\to\pm\infty$; (iii) $x''+cx'+kx=0$ has no solution $\not\equiv 0$ of period L; (iv) the unique solution of period L of $x''+cx'+kx=e(t)$ vanishes only on a set of measure zero. The case $k=0$, excluded because of condition (iii), is also treated by strengthening (ii) and suitably modifying (iv). These results imply that the equation can have no subharmonics of order n (i.e. solutions of period nL) provided that $A\geqq A(n)$; it is not asserted that the constant $A(n)$ can be chosen to be independent of n. *G. E. H. Reuter.*

Protasov, V. I. On a linear differential equation of infinite order. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 1189–1192. (Russian)

The author proves theorems such as the following. If $\lim\sup(|a_n|/n!)^{1/n}=R^{-1}$, $\lim\sup(n!|b_n|)^{1/n}<R$, then there exists a solution $y(x)=\sum_{k=0}^\infty c_k x^k/k!$ of the differential equation $\sum_{n=0}^\infty a_n y^{(n)}=\sum_{k=0}^\infty b_k x^k/k!$ with the property $\lim\sup(n!|c_n|)^{1/n}<R$. The proof is by reduction to an infinite system of linear equations for the c_k and uses properties of transposed matrix transformations in the dual of a Köthe-Toeplitz space. *G. G. Lorentz.*

Kreĭn, S. G. Differential equations in a Banach space and their application in hydromechanics. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 208–211. (Russian)

The author shows that a modified form of the Navier-Stokes equation for the flow of viscous fluids may be put in the form $dx/dt=Ax+f$, where x and f are functions of time with range in a certain Hilbert space and A is a negative-definite self-adjoint operator. He then announces that in conjunction with M. A. Krasnosel'skiĭ and P. E. Sobolevskiĭ a detailed study was made of the dependence of the solutions of this differential equation on f [cf. R. Phillips, Trans. Amer. Math. Soc. **74** (1953), 199–221; MR **14**, 882], which led to the solution of the modified Navier-Stokes equation by means of an integral. No details of the study are given. *A. Devinatz.*

See also: Sucheston, p. 21; Schubart und Wittich, p. 24; Yu, p. 28; Thorne, p. 28; Avdeev, p. 37; Štraus, p. 47; Trjitzinsky, p. 50.

Partial Differential Equations

Ehrenpreis, Leon. Sheaves and differential equations. Proc. Amer. Math. Soc. **7** (1956), 1131–1138.

Ce travail est une tentative d'application de la théorie des faisceaux [voir Séminaire H. Cartan de l'Ecole Norm. Sup., 1950/1951, Paris, 1955; MR **17**, 1117; Serre, Ann. of Math. (2) **61** (1955), 197–278; MR **16**, 953] à la recherche de solutions globales d'équations différentielles. L'auteur considère un sous-espace M d'une variété indéfiniment différentiable R et il donne une définition axiomatique très générale de „opérateur différentiel" D, admettant, comme cas particulier, celui des opérateurs aux dérivées à coefficients C^∞ sur M. Il considère d'abord les

faisceaux suivants sur M: le faisceau E des germes de fonctions C^∞, le faisceau DE défini d'une façon évidente et le faisceau A des espaces A_x ($x \in M$) formés des $f \in E_x$ telles que $Df=0$ autour de x. La discussion est basée sur les groupes de cohomologie $H^j(A)$ de M à coefficients dans A. On dit que D est fin, si $DE=E$ (opérateurs elliptiques ou hyperboliques, etc.). Si D est fin, alors $H^j(A)=0$ pour $j\geqq 2$; si, en outre, $\mathcal{E}$ désigne l'espace des fonctions C^∞ sur M, on a $H^1(A)\cong\mathcal{E}/D\mathcal{E}$. Des faisceaux de distributions sont aussi considérés.

Les problèmes aux limites donnent lieu à une formulation abstraite assez générale: la frontière est remplacée par un sous-ensemble distingué $\dot{M}$ de M, les „conditions aux limites" sont données par un faisceau K sur $\dot{M}$ et les solutions de $Df=0$ vérifiant ces conditions forment un faisceau G sur R. Un théorème général est établi, dont voici une conséquence: pour que toute „valeur admissible" sur $\dot{M}$ soit la restriction à $\dot{M}$ d'une solution globale de $Df=0$, il suffit que $H^1(G)=0$. D'autres résultats sont établis. *J. Sebastião e Silva* (Lisbonne).

Godunov, S. K. On uniqueness of the solution of hydrodynamic equations. Mat. Sb. N.S. **40(82)** (1956), 467–478. (Russian)

Es handelt sich um das mit den hydrodynamischen Gleichungen

$$(*)\quad \frac{\partial u}{\partial t}+\frac{\partial p(v,E)}{\partial x}=0,\quad \frac{\partial v}{\partial t}-\frac{\partial u}{\partial x}=0,$$
$$\frac{\partial(E+\frac{1}{2}u^2)}{\partial t}+\frac{\partial pu}{\partial x}=0$$

verknüpfte Eindeutigkeitsproblem. Systeme dieser Art gestatten bekanntlich auch im Falle stetig vorgegebener Anfangsbedingungen nicht immer in hinreichend großen Bereichen verlaufende Lösungen ohne Unstetigkeiten. Daher empfiehlt es sich in der Behandlung unstetiger Lösungen das System (*) durch die Integralbeziehungen

$$\oint_\Gamma udx-p(v,E)dt=0,\ \oint_\Gamma vdx+udt=0,\ \int_\Gamma (E+\tfrac{1}{2}u^2)dx-pudt=0$$

zu ersetzen, deren Integration über eine beliebige glatte Berandung Γ im Bereich der in Rede stehenden Lösung verläuft. Solche unstetige Lösungen wurden bereits von O. A. Oleĭnik [Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 451–454; MR **16**, 253] und A. N. Tychonoff und A. A. Samarski [ibid. **99** (1954), 27–30; MR **16**, 704] behandelt. Doch bleiben die Methoden dieser Autoren auf Einzelgleichungen beschränkt. — Verfasser zeigt, daß im Falle nichtlinearer Differentialgleichungen die gewöhnliche Methode auch bei Unstetigkeiten zum Beweis eines Eindeutigkeitssatzes führt, wenn man einen Grundgedanken eines Satzes von Holmgren verwendet, wie er auch bereits von I. G. Petrowski breit entwickelt worden ist. — In den §§ 1–3 wird dieser Beweis zunächts für den einfacheren Fall des Systems

$$\frac{\partial u}{\partial t}+\frac{\partial p(v)}{\partial x}=0,\quad \frac{\partial v}{\partial t}-\frac{\partial u}{\partial x}=0$$

durchgeführt, wie er z.B. in der Hydromechanik seichter Strömungen vorliegt. In § 4 wird gezeigt, wie eine ausführlichere Beweisführung nötig wird, wenn man zu den allgemeineren Systemen (*) übergehen will. Dabei ergibt sich ein Eindeutigkeitssatz für ein allgemeineres Differentialsystem der Gasdynamik. *M. Pinl* (Köln).

Aržanyh, I. S. Qualitative difference of holonomic from nonholonomic systems. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **10** (1953), no. 2, 179–185. (Russian)

Aržanyh, I. S. On an error in the theory of integration of the equations of motion of holonomic and nonholonomic systems. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **13** (1954), 163–167. (Russian)

Aržanyh, I. S. Extension of the potential method of integration to canonical involution systems in exact differentials. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **15** (1955), 87–91. (Russian)

Avdeev, N. Ya. On the question of solution of a mixed system of differential equations. Rostov. Gos. Ped. Inst. Uč. Zap. no. 3 (1955), 59–70. (Russian)

Systems of the type

$$M(x,y)dx+N(x,y)dy=0,$$
$$A\partial^2u/\partial x^2+2B\partial^2u/\partial x\partial y+C\partial^2u/\partial y^2+D\partial u/\partial x+E\partial u/\partial y=f(x,y)$$

are considered. A one parameter family of solutions is determined when $f=0$, $\partial M/\partial y=\partial N/\partial x$, and M and N satisfy a first order equation with constant coefficients (related to $A,\cdots,E$). If $f=0$, then a simple necessary and sufficient condition that the system have a solution is given. The inhomogeneous case is discussed. Numerous examples are given. *N. D. Kazarinoff.*

Mambriani, Antonio. La pluriderivazione e una classificazione delle equazioni differenziali. Riv. Mat. Univ. Parma **6** (1955), 321–348.

L'A. dà il nome di pluriderivatore a un operatore differenziale lineare del tipo: $D=\sum X_i\partial/\partial x_i$, dove le X_i sono funzioni delle variabili $(x_1, x_2, \cdots, x_n)$. Secondo un precedente risultato di B. Manfredi [Boll. Un. Mat. Ital. (3) **4** (1949), 381–390; MR **11**, 520] ogni pluriderivatore, nell'ipotesi $X_1\neq 0$, è suscettibile di una decomposizione del tipo:

$$D=X_1S^{-1}\frac{\partial}{\partial x_1}S,$$

dove S e S^{-1} sono due sostituzioni, inversa l'una dell'altra, che mutano le variabili $x_2, \cdots, x_n$ in certe funzioni di $x_1, \cdots, x_n$. Sulla base di tale formula l'A. studia le proprietà formali del calcolo con l'operatore D, con particolare riguardo alla costruzione dell'operatore inverso D^{-1}. L'algoritmo così istituito consente di utilizzare per la risoluzione di certte equazioni a derivate parziale artifici analoghi a quelli che si adoperano di solito per l'integrazione a mezzo di quadrature delle equazioni differenziali ordinarie. *C. Miranda* (Napoli).

Oleĭnik, O. A. The problem of Cauchy for non-linear differential equations of the first order with discontinuous initial conditions. Trudy Moskov. Mat. Obšč. **5** (1956), 433–454. (Russian)

Die vorliegende Arbeit enthält eine ausführliche Darstellung der Ergebnisse einer früheren Untersuchung [Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 451–454; MR **16**, 253] und darüber hinaus weitere Ergebnisse in der Theorie nichtlinearer partieller Differentialgleichungen erster Ordnung im Falle unstetiger Anfangsbedingungen, die sich aus einer modifizierten Formulierung des Cauchyschen Problems ergeben. Für unstetige Anfangsbedingungen kann eine Lösung des (klassischen) Cau-

chyschen Problems nicht eindeutig bestimmt werden, nicht einmal in einer beliebig kleinen Umgebung. Ähnliche Schwierigkeiten treten auf, wenn es sich um globale Probleme handelt (Existenz von Lösungen im Großen) und endlich können beide Schwierigkeiten auch kombiniert auftreten. Verfasser beschränkt seine Untersuchungen im Wesentlichen auf den Fall quasilinearer Differentialgleichungen erster Ordnung der Gestalt

$$(*)\quad f_1(t,x,u)\frac{\partial u}{\partial t}+f_2(t,x,u)\frac{\partial u}{\partial x}+f_3(t,x,u)=0,\qquad f_1(t,x,0)\neq 0,$$

deren Anfangsbedingungen im Intervall $[a, b]$ der Geraden $t=0$, bzw. für alle Punkte dieser Geraden gestellt werden. Die Frage nach dem Zusammenhang der hier erzielten Ergebnisse mit denen, die das Cauchysche Problem und das erste Randwertproblem der parabolischen Differentialgleichungen

$$\varepsilon\frac{\partial^2 u}{\partial x^2}=f_1(t,x,u)\frac{\partial u}{\partial t}+f_2(t,x,u)\frac{\partial u}{\partial x}+f_3(t,x,u)$$

für kleine Werte $\varepsilon>0$ betreffen, soll in einer späteren Untersuchung behandelt werden [ibid. **109** (1956), 1098–1101; MR **18**, 656].

Zunächst kann an Stelle der Differentialgleichung (*) die Differentialgleichung

$$(**)\qquad \frac{\partial u}{\partial t}+\frac{\partial\varphi(t,x,u)}{\partial x}=0$$

zu Grunde gelegt werden, welche bereits die charakteristischen Ergebnisse der zu entwickelnden Theorie liefert. Als Grundgebiet G wird jener Teil der Halbebene $t>0$ benutzt, der vom Intervall $[a, b]$ auf $t=0$ und den Projektionen der beiden Charakteristiken der Gleichung (**) begrenzt wird, die durch die Punkte $(0, a, \lim\sup_{x\to a}u_0(x))$ und $(0, b, \lim\inf_{x\to b} u_0(x))$ gehen. Dabei sind diese Charakteristiken Lösungen des Systems

$$\frac{dx}{dt}=\varphi_u'(t,x,u),\quad \frac{du}{dt}=-\varphi_x'x(t,x,u)$$

mit den Anfangsbedingungen $x=a$, $u=\lim\sup_{x\to a}u_0(x)$ für $t=0$, $x=b$, $u=\lim\inf_{x\to b}u_0(x)$ für $t=0$. Nach eingehender weiteren Untersuchung der notwendigen Voraussetzungen über die Funktion $\varphi(t, x, u)$ wird die angekündigte Neuformulierung des Cauchyschen Problems (für die Gleichung (**)) folgendermaßen gegeben; die Funktion $u(t, x)$ heißt eine Lösung des Cauchyschen Problems der Gleichung (**) im Gebiet G mit den Anfangsbedingungen $u(0, x)=u_0(x)$ innerhalb $a\leq x\leq b$, wenn die folgenden Bedingungen erfüllt sind: (1) wenn die Funktion $u(t, x)$ im Punkt (t_1, x_1) in G stetig ist, so gehört die Charakteristik durch den Punkt $(t, x_1, u(t_1, x_1))$ zur Lösung $u(t, x)$; (2) wenn die Funktion $u(t, x)$ in (t_1, x_1) unstetig ist, so soll es mindestens zwei Charakteristiken geben derart, daß für $0<t<t_1$ alle ihre Punkte zur Lösung $u(t,x)$ gehören und die Projektionen dieser Charakteristiken durch den Punkt (t_1, x_1) gehen; (3) das Linienintegral

$$\int\varphi(t,x,u(t,x))dt-u(t,x)dx$$

verschwindet, wenn die Integration über den Weg erstreckt wird, der von den Projektionen zweier beliebiger Charakteristiken gebildet wird, wie sie in (2) angegeben worden sind, und der Geraden $t=t_0$, wobei $0\leq t_0\leq\gamma$ gilt und γ eine gewissen positive Zahl ist; (4) die Funktion $u(t, x)$ ist im Gebiet G beschränkt und für jeden festen Wert t bezüglich x integrierbar. Für ein beliebiges Intervall $[x_1, x_2]$ innerhalb $[a, b]$ gilt

$$\lim_{\delta\to 0}\left|\int_{x_1}^{x_2}(u_0(x)-u(\delta,x))dx\right|=0.$$

Dann wird gezeigt, daß für geeignete Funktionen $\varphi(t, x, u)$ und $u_0(x)$ im Gebiet G eine eindeutige Lösung des neuformulierten Cauchyschen Problems mit den Anfangsbedingungen $u(0, x)=u_0(x)$ existiert. Zur näheren Untersuchung der Eigenschaften dieser Lösung beweist Verfasser nicht weniger als 11 Hilfssätze und 9 Sätze. Als Hauptergebnis (Satz 9) sei erwähnt: Die Lösung des Cauchyschen Problems der Gleichung (**) existiert mit den Anfangsbedingungen $u|_{t=0}=u_0(x)$ in der Halbebene $t\geqq 0$ eindeutig, wenn $|\varphi_x'(t, x, 0)|\leq K$, $-\varphi_{xu}'\leq C$ ist, wobei C und K gewisse Konstanten bedeuten, und

$$|\varphi_u'(t,x,u)|<M$$

gilt, für alle x, solange t und u in beschränkten Gebieten variieren. *M. Pinl* (Köln).

Henrici, Peter. A survey of I. N. Vekua's theory of elliptic partial differential equations with analytic coefficients. Z. Angew. Math. Phys. **8** (1957), 169–203.

In a series of papers which have appeared since 1937 and which were summarized in a book "New methods for solving elliptic equations" [OGIZ, Moscow-Leningrad, 1948; MR **11**, 598], I. N. Vekua developed a theory of linear elliptic partial differential equations with analytic coefficients and with two independent variables in a way which generalizes some aspects of the classical theory of holomorphic functions. This book received a brief notice in MR **11**, 598, and has never been translated into English.

The purpose of the present paper is to give an introduction to Vekua's theory. It includes some original material which well illustrates the usefulness of the theory.

It should be noted that, in the earlier volumes of these Reviews, Vekua's surname is spelt Vecoua.

E. T. Copson (Cambridge, Mass.).

Koschelev, A. I. On the boundedness in L_p of the highest derivatives of solutions of elliptic differential equations. Vestnik Leningrad. Univ. **12** (1957), no. 1, 165–167, 211. (Russian. English summary)

Let Ω be a bounded domain of n-space whose boundary Γ is composed of manifolds, and let a linear elliptic operator of second order be defined on Ω. Assume also that the Dirichlet problem with zero boundary conditions has at most one solution. Let $W_p^2(\Omega)$ be the space of functions on Ω with generalized second order derivatives belonging to $L_p(\Omega)$. This paper is devoted to the theorem that if $p>1$, and if the coefficients of the elliptic operator and the boundary Γ satisfy certain regularity conditions (which depend on the value of p), then the inverse of the operator exists and is continuous as a transformation from $L_p(\Omega)$ to $W_p^2(\Omega)$. There are several typographical errors, and it seems to this reviewer that the proof is somewhat incomplete. *G. Hufford* (Stanford, Calif.).

Cordes, Heinz Otto. Nicht-halbbeschränkte partielle Differentialoperatoren bei Randbedingungen dritter Art. Math. Nachr. **15** (1956), 240–249.

This paper is concerned with differential operators of the form $D=-(\partial^2/\partial x^2+\partial^2/\partial y^2)=-\Delta$ in the Hilbert space $\mathfrak{H}$ of all square integrable functions on a bounded region G of the (x, y)-plane, with boundary Γ a Jordan curve

having continuously turning tangent and piecewise continuous curvature. The domain $\mathfrak{D}$ of D consists of functions $u(x, y)$ such that: (a) u is of class C' on $G+\Gamma$, and in each closed subdomain of G has piecewise continuous second order partial derivatives; (b) $\Delta u \in \mathfrak{H}$; (c) $\partial u/\partial n-\sigma(x, y)u=0$ on Γ, where $\sigma(x, y)$ is continuous at points of Γ with the possible exception of a given point (x_0, y_0); (d) $u\equiv 0$ in a neighborhood of (x_0, y_0). The author shows that if the order of the singularity of σ at (x_0, y_0) is less than 1 then D, and each of its hermitian extensions, is bounded below, whereas for a presented class of functions σ for which the order of this singularity is greater than 1 the operator is not bounded below. The method of proof involves the application of a result on conformal mapping to obtain these results for general regions from the results for a particular region whose boundary contains the point of singularity as an interior point of a linear interval. *W. T. Reid.*

Friedman, Avner. On n-metaharmonic functions and harmonic functions of infinite order. Proc. Amer. Math. Soc. 8 (1957), 223–229.

L'auteur appelle fonction n-métaharmonique dans un domaine de l'espace euclidien E^N à N dim. une fonction satisfaisant à l'équation écrite symboliquement

$$p(\Delta)u(x)=0,$$

où $\Delta=\sum \partial^2/\partial x_i^2$ et $p(t)$ un polynome de degré n qu'on supposera à coefficients réels. On considère les solutions u dans un domaine contenant la fermeture d'un domaine D ayant une certaine disposition par rapport à l'axe des x_1. On montre que chaque u s'exprime dans D de manière simple au moyen de diverses solutions d'équations du type $(\Delta-\gamma)v(x)=0$. On en déduit que si p n'a pas de racines <0, toute solution dans E^N, bornée, est constante; on conclut même sans hypothèse sur les racines en remplaçant la condition de majoration par une certaine régularité à l'infini. Puis, si en outre $p(0)\neq 0$, on montre qu'une certaine limitation de croissance à l'infini implique $u=0$. Enfin on examine les fonctions dites harmoniques d'ordre infini selon Aronszajn, c'est à dire telles que $n^{-2}|\Delta^n u|^{1/n}$ tende vers 0 uniformément localement. Dans E^N, si de plus $|\Delta^n u|^{1/n}$ tende vers 0 en tout point, la fonction u est constante si elle est bornée. *M. Brelot.*

Duff, George F. D. Eigenvalues and maximal domains for a quasilinear elliptic equation. Math. Ann. 131 (1956), 28–37.

Let D be a bounded domain with boundary B in a Riemannian space V with positive definite metric tensor a_{ik} which is analytic in the coordinate system

$$x=(x_1, x_2, \cdots, x_N).$$

The differential equation (1) $\Delta u=|\text{grad } u|^2+\lambda\rho(x)$ is considered, where Δ is the Laplacian operator in V and $|\text{grad } u|^2=a^{ik}(\partial u/\partial x_i)(\partial u/\partial x_k)$ and λ is a parameter. By the transformation $\varphi=e^{-u}$, the equation becomes (2) $\Delta\varphi+\lambda\rho\varphi=0$ which is linear, homogeneous and elliptic for φ. Any solution of (1) is a positive solution of (2) and conversely. Let $\varphi(x)>0$ and λ_1 be the first eigenvalue of (2) for the Dirichlet problem. It is proved that 1) for $\lambda<\lambda_1$, there exists an unique solution of (1) for the Dirichlet problem on D, and 2) for $\lambda\geqq\lambda_1$ there exists no solution regular on $D+B$. By keeping $\lambda<\lambda_1$, D becomes a maximal domain for the Dirichlet problem of (1) in the sense that for these λ any larger domain containing D will admit no solution. The Neumann and Stekloff problems for (1) become problems for (2) with homogeneous boundary conditions, and in each case the solution exists only for an unique value of λ. The considerations are extended to the equations of the form (3) $\Delta u=K(u)|\text{grad } u|^2+\lambda\sigma(x, u)$ with the additional assumption that both σ and $\int_0^u K(s)ds$ are bounded for all u, $-\infty<u<\infty$. A suitable transformation $u=y(v)$ brings the equation to (4) $\Delta v=\lambda F(x, v)$ where F is bounded for all v and all x in $D+B$. It is shown by the method of Schauder-Leray that for the Dirichlet problem with differentiable boundary values (4) and hence (3) has a solution. *Y. W. Chen* (Detroit, Mich.).

Payne, L. E.; and Weinberger, H. F. Remark on a paper of O. G. Owens. Duke Math. J. 24 (1957), 233.

O. G. Owens [same J. 23 (1956), 371–383; MR 18, 132] showed that the solution of $u_{tt}=u_{xx}+u_{yy}$ taking the values of a homogeneous polynomial on the characteristic cone $t^2=x^2+y^2$ is itself a homogeneous polynomial in (x, y, t). A much simpler proof is given in this note; it depends on Maxwell's formula for spherical harmonics. *E. T. Copson* (Cambridge, Mass.).

Samosiuck, G. P. On Goursat's periodic problem. Vestnik Leningrad. Univ. 12 (1957), no. 1, 97–109, 210. (Russian. English summary)

Sommerfeld's method of solving the diffraction problem, by introducing a "Riemann" space in which wave functions that are many-valued in ordinary space become single-valued, is applied to a Goursat problem for the wave equation $\Delta u=u_{tt}$ in three spatial dimensions: to find a solution $u(r, \theta, \varphi, t)$, periodic in the azimuthal variable θ, which assumes the values of a given periodic function on the characteristic surface $r=t$. The required function is first obtained as a formal analogue of the solution resulting from separation of variables in the non-periodic case. It is then established rigorously as the solution to the problem in question. *R. N. Goss.*

Szmydt, Z. Sur un nouveau type de problèmes pour un système d'équations différentielles hyperboliques du second ordre à deux variables indépendantes. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 67–72.

The author considers for a vector function $U(x, y)$ in the rectangle $0\leqq x\leqq\alpha$, $0\leqq y\leqq\beta$, a system of hyperbolic differential equations $U_{xy}=F(x, y, U, U_x, U_y)$. F is continuous for $0\leqq x\leqq\alpha$, $0\leqq y\leqq\beta$ and $|U|$, $|U_x|$, $|U_y|\leqq r$, where $0<r<\infty$. Moreover, F satisfies a hypothesis K introduced in this paper to replace the customary stronger requirement of Lipschitz condition with respect to the arguments U_x and U_y in F. Two curves Γ $(y=\gamma(x))$ and Λ $(x=\lambda(y))$ are given in the rectangle; they are not necessarily distinct. Problem I requires $U_x=G(x, U, U_y)$ on Γ, $U_y=H(y, U, U_x)$ on Λ and $U(x_0, y_0)=U_0$ at a given point (x_0, y_0). Problem II requires the same on Γ and $U=B(y)$ on Λ. The constant U_0 and the continuous functions G, H, B are given in the above problems. It is stated that for each problem the existence of at least one solution is proved by using Schauder's fixpoint theorem. The detailed assumptions needed for the proofs are formulated for problem II and a special case of I. These conditions are concerned with the "sizes" of the given functions; for instance: let $h=\max(\alpha, \beta)$, then

$$N=\max F\leqq\min(r/h, r/2h^2)$$

and $|U_0|$, $|G|$, $|H|\leqq\min\{r-hN, (r-2Nh^2)/(1+2h)\}$ for the case that G and H are independent of U, U_x and U_y. *Y. W. Chen* (Detroit, Mich.).

Artemow, G. A. Die Anwendung der S. A. Tschaplygin-Methode zur Lösung der Cauchy Aufgabe für nicht lineare Gleichungen in partiellen Ableitungen der zweiten Ordnung des hyperbolischen Typs. Ukrain. Mat. Ž. 9 (1957), 5–19. (Russian. German summary)

Čaplygin's method consists in the construction of sequences of functions, based on a differential inequality, which converge to the solution of a given boundary-value problem. In this paper the relevant inequality is established for the equation $u_{xy}=f(x, y, u, p, q)$, $(p=u_x, q=u_y)$, with initial conditions $u=u(t)$, $p=p(t)$, $q=q(t)$ along a given non-characteristic curve. Sequences of functions which approximate u from above and below are constructed, successive members being related by a recurrence formula containing the Riemann function associated with the problem. Convergence to the solution is proved, and rapidity of convergence is estimated.
R. N. Goss (San Diego, Calif.).

Guss', G.; Poenaru, V.; and Foyaš, K. Direct method in the Cauchy problem for a quasi-linear hyperbolic equation involving two independent variables. Dokl. Akad. Nauk SSSR (N.S.) 112 (1957), 381–382. (Russian)

The authors announce theorems on existence, uniqueness, correctness of set, continuation, boundedness, and stability of the generalized (Sobolev) solution of the equation $u_{xx}-u_{yy}=f(x, u, y, u_x, u_y)$ with Cauchy data for $y=0$, obtained by a method due to P. Montel [Ann. Sci. Ecole Norm. Sup. (3) 24 (1907), 233–334, Ch. 2]. Proofs are not given. *R. N. Goss* (San Diego, Calif.).

Krzyżański, M. Sur l'allure asymptotique des potentiels de chaleur et de l'intégrale de Fourier-Poisson. Ann. Polon. Math. 3 (1957), 288–299.

The author calls attention to an analogy between certain integrals involved in the solutions of classical boundary value problems for the equation $\partial^2u/\partial x^2=\partial u/\partial t$ and the potentials due to simple and double distributions. The paper contains an investigation of the asymptotic behavior (as $t\to\infty$) of these integrals and the associated solutions. One theorem, for example, states that if $\varphi(s)$ is measurable and bounded and if

$$u(x, t)=\frac{1}{2\sqrt{(\pi t)}}\int_0^\infty \varphi(s)\left\{\exp\left[-\frac{(x-s)^2}{4t}\right]-\exp\left[\frac{(x+s)^2}{4t}\right]\right\}ds$$

(so that $\partial^2u/\partial x^2=\partial u/\partial t$, $t>0$; $u(x, 0)=\varphi(x)$, $x\geqq 0$; $u(0, t)=0$ $t\geqq 0$) then, for any positive R, $\lim_{t\to\infty} u(x, t)=0$ uniformly with respect to x on $0\leqq x\leqq R$. *F. W. Perkins.*

Kim, E. I. On a plane problem of the heat-conduction equation. Rostov. Gos. Ped. Inst. Uč. Zap. 1953, no. 2, 31–38. (Russian)

This paper considers the equation of heat conduction

$$(1)\qquad \frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

in the domains D_1 and D_2 for $0<t\leqq T$. Here D_1 is the domain contained inside a closed curve c with continuous curvature lying above the x-axis, and D_2 a domain which consists of the upper half plane excluding D_1. The initial and boundary conditions are

$$(2)\qquad u|_{t=0}=f(x, y), \qquad (3)\qquad u|_{y=0}=\phi(t, x),$$

$$(4)\qquad K_1\frac{\partial u}{\partial n}=K_2\frac{\partial u}{\partial n} \text{ on the curve } c,$$

where f and ϕ are given continuous bounded functions such that $f(x, 0)=\phi(0, x)$. The solution u is expressed in terms of integrals involving two additional functions which are the solutions of a system of two singular integral equations. The solutions of the integral equations are given in a power series of a certain parameter λ which depends upon K_1 and K_2. *C. G. Maple* (Ames, Iowa).

Hille, Einar. Quelques remarques sur l'équation de la chaleur. Rend. Mat. e Appl. (5) 15 (1956), 102–118.

Let C_ρ $(\rho\geqq 0)$ be the space of all continuous functions $f(x)$ on $(-\infty, \infty)$ such that $f(x)\exp(-|x|^\rho)$ tends to finite limits as x tends to $\pm\infty$. C_ρ is a Banach space by the norm $\|f\|_\rho=\sup_x|f(x)|\exp(-|x|^\rho)$. An abstract Cauchy problem (PAC), in the author's sense [see J. Analyse Math. 3 (1954), 81–196; MR 16, 45], for the equation

$$\partial u/\partial t=\partial^2u/\partial x^2$$

is discussed in C_ρ, showing that the situations are entirely different according to the three cases, 1) $0\leqq\rho\leqq 1$, 2) $1<\rho\leqq 2$, and 3) $2<\rho$: 1) For every initial function $f\in C_\rho$, the classical solution

$$W(x, t, f)=\int_{-\infty}^{\infty} 2^{-1}(\pi t)^{-\frac{1}{2}}\exp(-s^2/4t)f(s+x)ds$$

gives a unique solution of PAC. 2) $W(x, t, f)$ gives a unique solution of PAC if $W(x, t, f)\in C_\rho$ for every $t>0$. However, there exists an $f\in C_\rho$ such that PAC does not admit a solution. 3) A positive increasing function $f(x)$ with $\lim_{x\to\infty}\exp(-x_\rho)f(x)>0$ does not give a solution which is positive increasing in $0<t<c$, $c>0$. In 3) there exists a solution $N(x, t)\not\equiv 0$ satisfying $\lim_{t\to 0}\|N(x, t)\|_\rho=0$, while such solutions do not exist in 1) and 2).
K. Yosida (Tokyo).

Kampé de Fériet, Joseph. Intégrales aléatoires de l'équation de la diffusion. C. R. Acad. Sci. Paris 243 (1956), 929–932.

Let $f(x)\geqq 0$ and $\exp(-cx^2)f(x)\in L_1(-\infty, \infty)$ with a $c\geqq 0$. Then, for any constant u,

$$s(x, t, u)=$$
$$\int_{-\infty}^{\infty}(4\pi t)^{-\frac{1}{2}}\exp(-(x-\xi)^2/4t)\exp(2^{-1}u(x-\xi)-4^{-1}u^2t)f(\xi)d\xi$$

is a solution of $\partial s/\partial t-\partial^2s/\partial x^2=-\partial(us)/\partial x$ in the band $0<t<\frac{1}{4}c$ satisfying the initial condition $\lim_{t\to 0} s(x, t, u)=f(x)$ a.e. If $u=u(\omega)$ is a normally distributed random variable such that $\bar u=0$, $\bar{u}^2=\sigma^2$, then the mean value $\bar s(x, t)$ of $s(x, t, u(\omega))$ cannot be defined except in the band $0<t<(1/\sigma^2)(\sqrt{(1+(\sigma^2/2c))}-1)$. In the case $c=0$, the moments of higher order of $s(x, t, u(\omega))$ exist in the band $t>0$. *K. Yosida* (Tokyo).

Morawetz, Cathleen S. Uniqueness for the analogue of the Neumann problem for mixed equations. Michigan Math. J. 4 (1957), 5–14.

Il s'agit d'une équation de la forme $K(\sigma)\omega_{\theta\theta}+\omega_{\sigma\sigma}=0$ dont on cherche une solution dans le domaine D limité par les arcs C_0, C_1, C_2, le long desquels la dérivée oblique $\omega_\sigma d_y/ds-K_\theta d\sigma/ds$ est connue, et par deux arcs de caractéristique issus d'un point situé sur la ligne $\sigma=0$. $K(\sigma)$ a le signe de σ; C_0 est tracé dans le demi plan $\sigma\geqq 0$, C_1 et C_2 dans le demi plan $\sigma\leqq 0$, et le long de ces arcs on doit avoir $Kd\sigma^2+d\theta^2\geqq 0$.

L'unicité de la solution du problème ainsi posé n'est assurée que si certaines inégalités sont satisfaites aux points où C_1 et C_2 recontrent la ligne $\sigma=0$. Un contre

exemple très simple montre la nécessité de cette condition. La démonstration utilise la fonction auxiliaire ψ déjà introduite par l'auteur dans un précédent article [Proc. Roy. Soc. London. Ser. A. **236** (1956), 141–144; MR **18**, 133] fonction qui vérifie un principe du maximum dans le domaine elliptique. La partie la plus délicate de la démonstration consiste à prouver que ψ ne peut atteindre un maximum local aux points de $\sigma=0$ situés sur C_1 et C_2.
P. Germain (Paris).

Sokolov, Yu. D. On some particular solutions of Boussinesq's equation. Ukrain. Mat. Ž. **8** (1956), 54–58. (Russian)

This paper gives some particular solutions of the nonlinear differential equations

$$(1)\qquad \frac{\partial h}{\partial t}=\frac{a^2}{2}\left(\frac{\partial^2 h^2}{\partial x^2}+\frac{\partial^2 h^2}{\partial y^2}\right),$$

$$(1_1)\qquad \frac{\partial h}{\partial t}=\frac{k}{m}\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right),$$

$$(1_2)\qquad \frac{\partial h}{\partial t}=\frac{a^2}{2r}\frac{\partial}{\partial r}\left(r\frac{\partial h^2}{\partial r}\right).$$

In some instances, certain boundary conditions are satisfied.
C. G. Maple (Ames, Ia.).

Stone, A. P. Some properties of Wigner coefficients and hyperspherical harmonics. Proc. Cambridge Philos. Soc. **52** (1956), 424–430.

In a four-dimensional space let

$$L_{\mu\nu}=-i\left(x_\mu\frac{\partial}{\partial x_\nu}-x_\nu\frac{\partial}{\partial x_\mu}\right).$$

Then the vectors $\mathbf{L}=(L_{23}, L_{31}, L_{12})$ and $\mathbf{M}=(L_{14}, L_{24}, L_{34})$ can be found, and from them

$$\mathbf{K}_1=\tfrac{1}{2}(\mathbf{L}+\mathbf{M}) \text{ and } \mathbf{K}_2=\tfrac{1}{2}(\mathbf{L}-\mathbf{M}).$$

The vectors $\mathbf{K}_1$ and $\mathbf{K}_2$ satisfy the commutation rules for angular momentum operator, as does $\mathbf{L}$. The four-dimensional rotation group can, therefore, be equally well represented by the eigenfunctions of K_1^2, K_2^2, K_{1z} and K_{2z}, and by the eigenfunctions of K_1^2, L^2, K_{1z}, K_{2z}. Since $\mathbf{L}=\mathbf{K}_1+\mathbf{K}_2$, the relation between the two sets of eigenfunctions is expressible in terms of Clebsch-Gordan coefficients. It is, however, possible to solve the four-dimensional Laplace equation to obtain these two sets of eigenfunctions directly. By evaluation at particular angles, values of particular Clebsch-Gordan coefficients may be obtained in closed form.
H. Feshbach.

Shuleshko, P. A new method of solving eigenvalue problems. Austral. J. Appl. Sci. **8** (1957), 7–26.

The "method of moments" may be used to approximate to the solution of a partial differential equation as follows. The unknown function $w(x, y)$, $0\leq x\leq 1$, $0\leq y\leq 1$, is replaced by the approximation

$$w=X(x)Y(y),$$

where X is a known function chosen to satisfy boundary conditions at $x=0$, 1 and Y is to be determined later. This expression is substituted in the differential equation for w (and the boundary conditions) which is then multiplied by X and integrated with respect to x over (0, 1). This yields an ordinary differential equation and boundary conditions for Y.

This method is used for rectangular orthotropic plates with sides parallel to the axes of elastic symmetry and with direct stresses on the edges. It is pointed out that the eigenvalue equation corresponding to an orthotropic plate under biaxial loading may be identical with that for an isotropic plate under uniaxial, so that known numerical results may be adapted to new problems.
R. C. T. Smith (Armidale).

Altman, M. On Ritz's method. Bull. Acad. Polon. Sci. Cl.III. **5** (1957), 23–27, III–IV. (Russian summary)

The author gives an exposition of the following well-known facts. (a) The usual Rayleigh-Ritz method for the solution of the nonhomogeneous problem $Au=f$ is equivalent to approximating f by a finite number of functions in the least square sense in the norm $(A\varphi, \varphi)$. (A is assumed positive definite, self-adjoint, and bounded below.) (b) This approximation is equivalent to a projection. (c) If a lower bound for A is known, the error in the approximation of f leads to a bound for the error in the approximation of u.
H. F. Weinberger.

Altman, M. A simple practical method and a compact computing scheme for the solution of linear equations in Hilbert space. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 29–33, IV (Russian summary) (1 plate).

The author notes that the projection occurring in the preceding paper can be done by orthogonalizing the approximating functions. He uses the Schmidt process without the normalization.
H. F. Weinberger.

See also: Babenko, p. 20; Janet, p. 32; Nikitin, p. 87; Armstrong, p. 93.

Difference Equations, Functional Equations

Fort, Tomlinson. Partial linear difference equations. Amer. Math. Monthly **64** (1957), 161–167.

In abbreviated notation, the equations studied by the author are of the form

$$\sum\{k_{ab}(i, j)y(i+a, j+b):(a, b)\in P\}=R(i, j),\ (i, j)\in S,$$

where P is a given finite set of ordered pairs of integers, and $R(i, j)$, as well as $k_{ab}(i, j)$ for each $(a, b)\in P$, is a given function of (i, j) whose domain S is the set of all integral points in a fixed rectangle. Particular attention is paid to the case where P is the "five-point pattern" $\{(-1, 0), (0, -1), (1, 0), (0, 1), (0, 0)\}$. The domain of a solution y is to be the set $\{(i+a, j+b):(i, j)\in S;\ (a, b)\in P\}$. Auxiliary linear conditions, usually boundary conditions, are sometimes imposed. In this context, the author considers the concepts of fundamental domain, fundamental system of solutions, compatibility of a system, and Green's function.
W. Strodt (New York, N.Y.).

Herlestam, Tore. On differential-difference equations. Nordisk Mat. Tidskr. **5** (1957), 29–36, 64. (Norwegian. English summary)

A discussion of the initial value problem

$$y'(x)=y(x-1)\ (x\geq 1),\quad y(x)=f(x)\ (0\leq x\leq 1).$$

Aczél, J.; and Hosszú, M. On transformations with several parameters and operations in multidimensional spaces. Acta Math. Acad. Sci. Hungar. **7** (1956), 327–338. (Russian summary)

For an n-dimensional vector function $f(x, U)$, with

$x=x_1, \cdots, x_n$ and $U=u_1, \cdots, u_m$, the transformations $y=f(x, U)$ form a semigroup if $f[f(x, U), V]=f(x, W)$, where $W=U\circ V$ is a function of the $2m$ parameters U, V. The authors determine the general solution of the functional equation $f[f(x, U), V]=f(x, U\circ V)$ for a given law of composition $U\circ V$ and for $m\leqq n$ under suitable solvability hypotheses on the functional equation $y=f(x, U)$. For additive parameters — that is, for the functional equation $f[f(x, U), V]=f(x, U+V)$ — they solve the problem also for $m>n$. Finally, the authors give necessary and sufficient conditions on the operation $U\circ V$ in order that the solution of the functional equation

$$(U\circ V)\circ W=U\circ(V\circ W)$$

can be written in the form $U\circ V=G^{-1}[G(U)+G(V)]$, where $G(U)$ is a topological mapping of the m-dimensional space on itself with $G(E)=0$ for the unit element E of the operation $U\circ V$. *E. F. Beckenbach.*

See also: de Rham, p. 20.

Integral and Integrodifferential Equations

★ **Lalescu, Traian. Introducere la teoria ecuaţiilor integrale. [An introduction to the theory of integral equations.]** Editura Academiei Republicii Populare Romîne, 1956. 134 pp. 5.90 Lei.

A reprinting of the first edition [Bucarest 1911], with a bibliography taken from the French translation [Paris, 1912]. There are chapters on the Volterra and Fredholm equations, singular equations and equations of higher order.

★ **Petrovskiĭ, I. G. Lectures on the theory of integral equations.** Translated from the second revised Russian edition by H. Kamel and H. Komm. Graylock Press, Rochester, N.Y., 1957. vi+97 pp. Paper \$1,95; cloth \$2.95.

For the Russian original see MR 13, 467 and MR 14, 761.

Janoš, Ludvik. Eine Approximation des ersten Eigenwertes einer homogenen Integralgleichung durch ein lineares Funktional. Časopis Pěst. Mat. **81** (1956), 304–330. (Czech. Russian and German summaries)

The author studies the largest eigenvalue $\lambda[M]$ of the equation

$$\int_0^1 \Gamma(x, s)y(s)dM(s)=\lambda y(x),$$

where M is monotone non-decreasing and of total variation 1 on (0, 1), and Γ is continuous, symmetric, and positive definite. The following linear approximation to $\lambda[M]$ is considered: $\lambda[M]\sim\int_0 VdM$, with $V\in C(0, 1)$. For a given V, an index N_V of inaccuracy for the approximation independent of M is defined essentially as

$$N_V=\sup_M\left|1-\left(\int_0^1 VdM/\lambda[M]\right|\right.;$$

and it is shown that inf N_V for $V\in C(0, 1)$ is assumed for $V(x)=2\Gamma(x, x)/(1-\alpha)$, where $\alpha=\sup_M\int_0^1\Gamma(x, x)dM/\lambda[M]$. This V then provides the best linear approximation uniformly in M in the sense that the largest error is as small as possible. *V. E. Beneš* (Murray Hill, N.J.).

Mihlin, S. G. On the theory of multidimensional singular integral equations. Vestnik Leningrad. Univ. **11** (1956) no. 1, 3–24. (Russian)

This paper is concerned with the singular operator A given by:

$$A\mu(x)=a(x)\mu(x)+\int_{E_m}\frac{f(x, \theta)}{r^m}\mu(y)dy,$$

where the x and y are vectors in euclidean m-spaces E_m, θ is the angle between x and $x-y$, r is the distance between x and y, and the integral is taken in the sense of a principal value. The author gives improved proofs of his previous results on the regularizability of A and on the behaviour of A as an operator on L_2.

In the concluding paragraphs the author makes some comments on a paper by Calderón and Zygmund; [Trans. Amer. Math. Soc. **78** (1955), 209–224; MR **16**, 816]. Professor Zygmund informs me that there is actually an oversight on p. 213, line 12 of the paper, which is however not difficult to correct without changing the essential idea of the proof. The correction appears in Trans. Amer. Math. Soc. **84** (1957), 559–560 [MR **18**, 894]. Meanwhile a completely different proof of a more general result was published by Calderón and him [Amer. J. Math. **78** (1956), 289–309, especially p. 290, Th. 2; MR **18**, 894].

J. J. Kohn (Princeton, N.J.).

Karcivadze, I. N. On the singular integral operator with discontinuous coefficients. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 450–452. (Russian)

Consider the singular integral operator defined on the unit circle C by

$$S\phi=\alpha(t)\phi(t)+(\pi i)^{-1}\beta(t)\int_C\phi(\sigma)(\sigma-t)^{-1}d\sigma,$$

where $\alpha(t)$ and $\beta(t)$ belong to some class L_{p_1} $(p_1>1)$, and regard S as operating from L_p $(p\geqq p_2/(p_1-1))$ into L_{p_2}, where $p_2=pp_1/(p+p_1)\geqq 1$. A family of examples is constructed, showing that S need to be of "generalised Fredholm type" [in the sense used by the reviewer, Mat. Sb. N.S. **28(70)** (1951), 3–14; MR **13**, 46]. In fact S need not have a closed range. *F. V. Atkinson* (Canberra).

Maravall Casesnoves, Darío. New types of differential and integrodifferential equations. New oscillation phenomena. Rev. Acad. Ci. Madrid **50** (1956), 287–435. (Spanish)

The contents of the ten chapters of this paper may be described briefly as follows: I. Study of homogeneous integro-differential equations of the form

$$(*)\qquad L[y]\equiv\sum_{r=0}^n a_r y^{(r)}(t)+\sum_{s=0}^m\int_0^\infty K(\tau)y^{(s)}(t-\tau)d\tau=0,$$

in which the coefficients a_r are constants, and the kernel functions $K_s(\tau)$ admit Laplace transforms. II. Consideration of general solution of $L[y]=F(t)$, where the $K_s(\tau)$ are such that the Fourier integrals $\int_0^\infty K_s(\tau)\sin w\tau\, d\tau$ and $\int_0^\infty K_s(\tau)\cos w\tau\, d\tau$ are convergent, and $F(t)$ is representable as a Fourier integral $F(t)=\int_{-\infty}^\infty f(\sigma)e^{-it\sigma}d\sigma$, with a special study of oscillation phenomena for the case in which $F(t)=M\cos wt$. III. Extension of results of Chapters I, II to systems of integro-differential equations. IV. Treatment of the equation $L[y]=a(t)$, with the coefficients a_r now functions of t, by means of reduction to a Volterra integral equation of the second kind. V. Solution of equations of the form (*) in which the limits of integration 0, ∞ are replaced by a, b. VI. Appli-

cation of preceding results to the solution of certain types of integro-differential equations in partial derivatives. VII. Linear differential equations with variable coefficients, and involving non-integral derivatives. VIII. Some examples from geometry and mechanics whose analytic formulation present equations of the type considered in the preceding chapter. IX. Solution of certain linear differential equations with variable coefficients, and continuous in the indices of derivation, by reduction to a Volterra equation of the first kind. X. Solution of linear stochastic differential equations of order n. *W. T. Reid* (Evanston, Ill.).

See also: Mysovskih, p. 66; Kaliski and Nowacki, p. 82; Haskind, p. 86.

Calculus of Variations

Craig, Homer V. On extensors in the calculus of variations. Math. Mag. **30** (1957), 175–191.

The concept of extensors (tensors under the extended point transformations) has been considerably developed in differential geometry, mechanics, relativity through the works of H. V. Craig [MR **11**, 60, 543; **13**, 384; **14**, 689, 1016], A. Kawaguchi [MR **16**, 285], M. Kawaguchi [MR **14**, 690; **15**, 255], and others. In particular [see MR **11**, 60], the present author had shown that the formalism of the extensors could be applied to the direct derivation of the Lagrange equations of a conservative system. In the present paper the author generalizes this process and shows that the Euler equation for the typical problems of calculus of variations for curves and surfaces can be expressed and obtained directly through the formalism of the extensors (that is, instead of using integration by parts and the fundamental lemma of the calculus of variations). The author takes into consideration the problem of the stationary curves with fixed endpoints, the corresponding isoperimetric problem, and the problem of the stationary surfaces of the type of the disc with a given boundary curve. The interest of the paper consists in the established connection between the formal problem of the calculus of variations and the extensor and tensor technique. *L. Cesari* (Lafayette, Ind.).

Fleming, W. H.; and Young, L. C. Generalized surfaces with prescribed elementary boundary. Rend. Circ. Mat. Palermo (2) **5** (1956), 320–340 (1957).

A generalized surface is a linear form $L(f)$ on a set F of functions, admissible for parametric problems in the calculus of variations, such that $f \geq 0$ implies $L(f) \geq 0$. The g boundary λ of $L(f)$ is its restriction $\lambda(f)$ to the set $A \subset F$ of exact functions. Convergence of L_n to L is weak * convergence (i.e., $\lim L_n(f) = L(f)$ for every $f \in F$). The set of rims of a surface is defined in terms of concrete representations in a more classical fashion. Part 1 of this paper studies the relation between the rims and the g boundary of a generalized surface. For each $R > 0$, G_R is the set of generalized surfaces contained in the sphere $|x| \leq R$. For each boundary λ (i.e., linear form on A), $G(\lambda)$ is the set of generalized surfaces with boundary λ. $G_R(\lambda) = G(\lambda) \cap G_R$. For thin, oriented, disjoint, simple, closed curves $C_1, \cdots, C_s$, contained in $|x| \leq R$,

$$\Gamma_R(C_1, \cdots, C_s)$$

is a class of surfaces, with representations satisfying smoothness conditions defined in the paper, with $C_1, \cdots, C_s$ as rims. $\langle \Gamma_R(C_1, \cdots, C_s) \rangle$ is the convex closure of $\Gamma_R(C_1, \cdots, C_s)$ in the space of generalized surfaces. It is shown that every set $C_1, \cdots, C_s$ of rims corresponds to a generalized boundary λ. The following is then proved: For surfaces in 3-space, $G_R(\lambda) = \langle \Gamma_R(C_1, \cdots, C_s) \rangle$, for every $C_1, \cdots, C_s$, corresponding λ, and $R > R_0$, where $|x| \leq R_0$ contains $C_1, \cdots, C_s$. For higher dimensions, this is proven false. It then follows that: For surfaces in 3-space, for every $f_0, C_1, \cdots, C_s$, and corresponding λ, the infimum of $L(f_0)$ on the set of surfaces with rims $C_1, \cdots, C_s$ equals its infimum on the set of generalized surfaces with g boundary λ. For higher dimensions, this is proved false.

Part 2 relates these ideas to those of a previous paper by the authors [same Rend. (2) **5** (1956), 117–144; MR **18**, 503]. For example, if $L_0 \in G_R(\lambda)$, and its track satisfies $tL_0 = L_1 + \cdots + L_k + \cdots$, where each L_k has a Lipschitzian representation on a bounded Borel set, and if λ corresponds to rims $C_1, \cdots, C_s$ contained in $|x| \leq R$, then for every $\varepsilon > 0$ there is L_ε with rims $C_1, \cdots, C_s$ such that $\|L_0 - L_\varepsilon\| < \varepsilon$ and L_ε has the same track as a parametric surface belonging to $\Gamma_R(C_1, \cdots, C_s)$. *C. Goffman.*

★ **Hestenes, Magnus R. Hilbert space methods in variational theory and numerical analysis.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 229–236. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

This paper presents a clear account of the connections between the classical calculus of variations and the Hilbert space or variational treatment of partial differential equations. In addition to giving some new results, the author makes a valuable contribution for specialists in either of these two fields by making results in the other field more comprehensible to them. Thus, for example, the strong Legendre condition is translated into ellipticity [cf. K. O. Friedrichs, Comm. Pure Appl. Math. **6** (1953), 299–326; MR **15**, 430] and the weak Legendre condition into weak lower semi-continuity.

The first part of the paper sumarizes earlier work of the author [Pacific J. Math. **1** (1951), 525–581; MR **13**, 759]. In the second part the author presents a method for converting any (not necessarily quadratic) minimum problem for a line integral into a minimax principle. The method appears to be closely connected with that of K. O. Friedrichs [Nachr. Ges. Wiss. Gottingen. Math.-Phys. Kl. **1929**, 31–20] for obtaining a maximum principle from a minimum principle. The final section of the paper is devoted to an extremely condensed presentation of the conjugate gradient method of the author and E. Stiefel [J. Res. Nat. Bur. Standards **49** (1952), 409–436; MR **15**, 651]. *H. F. Weinberger* (Madison, Wis.).

See also: Sapa, p. 78.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

See: Hoffman and Singer, p. 46; Freudenthal, p. 54.

Lie Groups and Algebras

Tits, J. Sur la géométrie des R-espaces. J. Math. Pures Appl. (9) 36 (1957), 17–38.

The author's purpose in this paper is to set forth a system of axioms, analogous to those for projective geometry, for a synthetic study of semisimple Lie groups. To achieve this goal he introduces the concept of point, distinguished subspaces (the analogue of the concept of planes in projective geometry) and incidence relations. He then introduces a system of axioms which these objects must satisfy. The basic ideas for this are all contained in previous work of the author. [Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° **29** (1955), no. 3; MR **17**, 874]. The main new result of this paper that is the exceptional simple Lie group E_6 is completely determined by the axiom system introduced here. *L. Auslander* (Bloomington, Ind.).

Mostert, Paul S. On a compact Lie group acting on a manifold. Ann. of Math. (2) **65** (1957), 447–455.

The author proves a number of interesting theorems on the action of a compact connected Lie group G on a manifold M of dimension $n+1$, with at least one n-dimensional orbit. It is shown that the orbit space M/G must be a circle, or an open interval, or a closed interval, or a half open interval. To some extent the structure of the manifold M, and the action of G on M, are recovered. {A little more is asserted than is actually proven.} In particular, the author completely determines all two-dimensional manifolds which admit a compact connected Lie group operating non-trivially: they are the seven Klein spaces. In addition he determines all compact connected Lie which groups can operate effectively on a three-dimensional manifold, and he determines all those three-dimensional manifolds which admit a compact connected Lie group acting effectively so that there is a two-dimensional orbit.

There is an error in Theorem 3 and in part (ii) of Theorem 4 (which was pointed out by Roger Richardson and Hans Samelson); the manifold need not be a cartesian product, and need not be orientable, when the orbit space is a circle. This is corrected in an addendum which will appear shortly. Even if the manifold is a product and the orbit space is a circle, the group need not operate in the "obvious way". For example, $O^+(4)$ can operate in two essentially different ways on $S^3\times S^1$. Also, there are some errors in the homology group given for the three-dimensional manifolds. *A. Shields* (Ann Arbor, Mich.).

LeLong-Ferrand, Jacqueline. Application of Hilbert space methods to Lie groups acting on a differentiable manifold. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 249–252.

Using Hilbert space methods, the author states five theorems on Lie groups acting on a differentiable manifold. Proofs are promised in a subsequent publication. Examples of the theorems:

1) If $\mathscr{G}$ is a compact ν-dimensional group and if X_α ($\alpha=1, 2, \cdots, \nu$) denotes a basis of its algebra, there exists a constant k such that $f\in\mathscr{H}$ (=Hilbert space of square-integrable functions) and $\int f d\tau=0$ imply $\|f\|^2\leq k\sum\|X_\alpha f\|^2$, where $d\tau$ is the invariant measure.

2) An element X of the Lie algebra of a Lie group $\mathscr{G}$ defines a one-parameter compact subgroup of $\mathscr{G}$ if, and only if, there exists a constant k such that every function $f\in\mathscr{H}$, orthogonal to the space $\mathscr{H}'=\{\varphi;\varphi\in\mathscr{H}, X\varphi=0\}$, satisfies $\|f\|\leq k\|Xf\|$. *S. Chern* (Chicago, Ill.).

See also: Freudenthal, p. 54.

Topological Vector Spaces

★ **Kolmogorov, A. N.; and Fomin, S. V. Elements of the theory of functions and functional analysis. Vol. 1. Metric and normed spaces.** Translated from the first Russian edition by Leo F. Boron. Graylock Press, Rochester, N. Y., 1957. ix+129 pp. $3.95.

A translation of the book reviewed in MR **16**, 1122.

★ **Marinescu, G. Spatii vectoriale normate. [Normed vector spaces.]** Editura Academiei Republicii Populare Romîne, 1956. 293 pp. 12.50 Lei.

There are 14 chapters: normed vector spaces and Banach spaces, linear and multilinear operators, prolongation of operators, conjugate spaces, adjoint operators, differential and integral operators in Banach spaces, Banach algebras, completely continuous operators, ergodic theory, methods of approximation, particular normed vector spaces, examples of functionals and operators in various particular spaces, applications to the theory of integral equations, various applications.

Kasahara, Shouro. Quelques conditions pour la normabilité d'un espace localement convexe. Proc. Japan Acad. **32** (1956), 574–578.

Let E be a locally convex space, and $\mathscr{L}(E, E)$ the algebra of continuous, linear transformations from E into E, equipped with the topology of uniform convergence on some family of bounded subsets of E. A condition on $\mathscr{L}(E, E)$ which implies that E is normable has recently been obtained independently by the author [Math. Japon. **3** (1955), 111–116; MR **17**, 510], A. Blair [Proc. Amer. Math. Soc. **6** (1955), 209–210; MR **16**, 935], and J. H. Williamson [J. London Math. Soc. **31** (1956), 111–113; MR **17**, 1111]; another such condition was also obtained by the author [Proc. Japan. Acad. **32** (1956), 131–134; MR **17**, 1111]. In this paper, the author obtains some refinements of these conditions which are, however, too technical to be quoted here. *E. Michael.*

Vasilach, Serge. Sur le produit de composition des fonctions et distributions à support dans $R_+{}^n$, $n>1$. C. R. Acad. Sci. Paris **244** (1957), 34–35.

A continuation of the paper reviewed in MR **18**, 491.

See also: Pinsker, p. 7; Cordes, p. 38; Hestenes, p. 43; Willcox, p. 46; Berberian, p. 47; Sakai, p. 47; Kadison, p. 47; Vulih, p. 47; Vaĭnberg, p. 48; Lafleur, p. 94; Schmidt und Baumann, p. 97.

Banach Spaces, Banach Algebras, Hilbert Spaces

Solomjak, M. Z. On orthogonal bases in Banach space. Vestnik Leningrad. Univ. **12** (1957), no. 1, 27–36, 208. (Russian. English summary)

Let E be a separable Banach space, let $\{z_n\}$ be its basis. Then $\{z_n\}$ is said to be orthogonal if and only if

$$\|\sum_{k=1}^{n+m} c_k z_k\| \geq \|\sum_{k=1}^{n} c_k z_k\|$$

is true for arbitrary scalars $c_1, c_2, \cdots, c_n, \cdots, c_{n+m}$. The author introduces the concept of a constrained subspace which gives a criterion for existence of an orthogonal basis. In fact, the author proves that the system of Haar [J. Schauder, Math. Z. **28** (1928), 317–320] is an orthogonal basis in $L_p([0, 1])$ and in arbitrary separable Orlicz spaces. A linear subspace E_0 of E is said to be constrained if and only if there exists a weakly closed (author uses the term "regularly closed") subspace Ξ_0 of the conjugate of E having the following properties: 1) For each $f \in \Xi_0$, $\sup_{x \in E_0, \|x\|=1} |f(x)| = \|f\|$; 2) for each $x \in E_0$ there exists $f \in \Xi_0$ such that $\|f\|=1$ and $f(x)=\|x\|$. The central theorem of the paper: E_0 is constrained if and only if there exists a projection P of E onto E_0 such that $\|P\|=1$. *P. Saworotnow* (Washington, D.C.).

Gohberg, I. C.; and Markus, A. S. On stability of certain properties of normally soluble operators. Mat. Sb. N.S. **40**(**82**) (1956), 453–466. (Russian)

Let A be an additive, closed, normally soluble operator on a Banach space, with $\alpha(A)$, $\beta(A)$ finite; known conditions on B in order that $A+B$ have the same properties, with the same value of the index, are that B be sufficiently small in norm, or else completely continuous [cf. papers of the reviewer, Mat. Sb. N.S. **28**(**70**) (1951), 3–14; MR **13**, 46; B. Sz.-Nagy, Acta Math. Acad. Sci. Hungar. 3 (1952), 49–52; MR **14**, 564; Gohberg, Mat. Sb. N.S. **33**(**75**) (1953), 193–198; MR **15**, 233]. The authors prove extensions in which not B but one of its iterates satisfies one of the latter restrictions. However a supplementary restriction has to be made, that $MB-BM$ is completely continuous, where M is a "regularising operator" for A, such that $MA-I$ is completely continuous. Next follow two rather less complete results on the stability of $m(A)$, the number of infinite rows in the canonical arrangement of null-elements of A [cf. S. N. Kračkovskiĭ, Dokl. Akad. Nauk SSSR (N.S.) **88** (1953), 201–204; MR **14**, 1095]. The cases considered are those in which $\alpha(A)=\beta(A)$, $m(A)=0$ and in which $m(A)=\alpha(A)<\infty$; in the first of these it is B^n which is restricted, in the latter it is B, in both cases B commuting with A. Finally, for the case $m(A)=\alpha(A)<\infty$, there is a result on the stability of the kernel-manifold of A, the intersection of the ranges of $A, A^2, \cdots$. *F. V. Atkinson* (Canberra).

Gohberg, I. C. On the index, null elements and elements of the kernel of an unbounded operator. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 177–179. (Russian)

Introductory notes on the subject-matter of the paper reviewed. Results of a forthcoming paper of the author with M. G. Kreĭn above are indicated. *F. V. Atkinson.*

Ladyženskiĭ, L. A. On non-linear equations with positive non-linearities. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 211–212. (Russian)

Expository article concerning work of the author and collaborators [Ladyženskiĭ, Dokl. Akad. Nauk SSSR (N.S.) **96** (1954), 1105–1108; Krasnosel'skiĭ and Ladyženskiĭ, Trudy Moskov. Mat. Obšč. **3** (1954), 307–320, 321–346; MR **16**, 48; **15**, 966]. *B. Gelbaum.*

Špaček, Antonín. Sur l'inversion des transformations aléatoires presque sûrement linéaires. Acta Math. Acad. Sci. Hungar. **7** (1956), 355–358. (Russian summary)

Notations: X a separable Banach space; $(F, \mathscr{F}, \mu)$ a probability space whose elements are transformations of X into itself; for a subspace A of X, $T(A)=\{f \mid f \in F,\ f$ is linear on $A\}$; $(F, \mathscr{F}, \mu)$ is called almost certainly linear if $\bar{\mu}(T(X))=1$ ($\bar{\mu}$ is outer measure engendered by μ); $t(f)(x) \equiv x-f(x)$.

Results: (1) $(F, \mathscr{F}, \mu)$ is almost certainly linear if and only if there is a constant C such that $\mu\{f \mid \|f\|(x)\| \leq C\|x\|\}=1$ for all x. (2) Let $(F, \mathscr{F}, \mu)$ be almost certainly linear, with $C<1$. Let $M \subset T(X)$, $\bar{\mu}(M)=1$, $\mathscr{U}$ a neighborhood system on $M \cap \mathscr{F}$. For fixed x let $f(x)$ be continuous in f for the given topology. Then the function $S(f)=[t(f)]^{-1}$ is almost certainly measurable, i.e., $S^{-1}(E) \in M \cap \mathscr{F}$ for all $E \in \mathscr{F}$. The proof depends completely on the continuity hypothesis and the relationship of the topological and measure structures. *B. Gelbaum* (Minneapolis, Minn.).

Rutickiĭ, Ya. B. On a class of Banach spaces. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 230–234. (Russian)

The discussion centers on Orlicz spaces for which the associated pair M, N of conjugate Young functions do not satisfy the well-known Δ_2 condition. General statements are made concerning the importance of the space E_M, the closure in L_M^* of the bounded functions of L_M^*. A restricted weak convergence is defined relative to E_N. Finally, it is observed: (a) that the construction of the conjugate space of L_M has not yet been accomplished; (b) that the construction of an N conjugate to a given M has been accomplished only in special cases; (c) that for the class of functions M of the form

$$u^r(\ln u)^{\alpha_1}(\ln\ln u)^{\alpha_2}\cdots(\ln\ln\cdots\ln u)^{\alpha_n}$$

(here, $r>1$; $\alpha_1, \alpha_2, \cdots, \alpha_n$ arbitrary) the construction of an $\tilde{N}$ for which the sets $L_{\tilde{N}}^*$ and L_N^* (N the unknown conjugate of M) are identical has been accomplished. Such an $\tilde{N}$ is called equivalent to N; it is given by

$$\tilde{N}(v)=v^{r/(r-1)}[(\ln v)^{\alpha_1}(\ln\ln v)^{\alpha_2}\cdots(\ln\ln\cdots\ln v)^{\alpha_n}]^{-1/(r-1)}.$$

B. Gelbaum (Minneapolis, Minn.).

Krasnosel'skiĭ, M. A. Study of the spectrum of a nonlinear operator in the neighborhood of a point of bifurcation and applications to the problem of longitudinal bending of a compressed rod. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 203–208. (Russian)

This is a somewhat short description of the relevance of the bifurcation points of the nonlinear operator of beam loading to the physical situation under discussion. *B. Gelbaum* (Minneapolis, Minn.).

Rooney, P. G. On some properties of certain fractional integrals. Trans. Roy. Soc. Canada. Sect. III. (3) **50** (1956), 61–70.

The author gives generalizations of theorems proved in his previous paper [same Trans. **49** (1955), 59–66; MR **17**, 646]. The fractional integral operators of Kober [Quart. J. Math. Oxford Ser. **11** (1940), 193–211; MR **2**, 191] are

defined by:

$$\mathfrak{J}_{r,s} f(x)=\frac{x^{-r-s}}{\Gamma(r)}\int_0^x (x-t)^{r-1}t^s f(t)dt,$$

$$\mathfrak{K}_{r,s} f(x)=\frac{x^s}{\Gamma(r)}\int_x^\infty (t-x)^{r-1}t^{-r-s}f(t)dt.$$

It is shown here that for appropriate choices of r, s, α, p in each case, $\mathfrak{J}_{r,s}$, $\mathfrak{K}_{r,s}$ are bounded linear operators over the spaces $\Lambda(\alpha, p)$, $M(\alpha, p)$ of G. G. Lorentz [Bernstein polynomials, Univ. of Toronto Press, 1953; MR 15, 217]. Upper bounds on the norms of the operators are given in each case. The paper concludes with applications to certain integral transforms of Whittaker functions.

R. E. Fullerton (College Park, Md.).

Hartman, Philip. On Laurent operators on l_p. Proc. Amer. Math. Soc. 8 (1957), 45–48.

A generalization of a theorem of Krabbe [same Proc. 7 (1956), 783–790; MR 18, 587] on the spectra of Laurent operators is proved. Let $f(\theta)$ be a function of bounded variation over $[-\pi, \pi]$ with Fourier coefficients $\{f_n\}$ $(n=0, \pm 1, \pm 2, \cdots)$, and define the Laurent operator $F(f):l_p\to l_p$ $(p>1)$ by the infinite matrix $\{f_{n-k}\}$. It is shown that the spectrum of $F(f)$ coincides with the closure of the range of f.

R. E. Fullerton.

Rudin, Walter. Continuous functions on compact spaces without perfect subsets. Proc. Amer. Math. Soc. 8 (1957), 39–42.

Let Q be a compact Hausdorff space which contains no perfect subset and let $C(Q)$ be the Banach algebra (in the sup norm) of all continuous complex valued functions on Q. The principal result is that every closed subalgebra of $C(Q)$ is self-adjoint. Consequently, every closed proper subalgebra of $C(Q)$ which separates the points of Q is a maximal ideal.

If Q is a compact Hausdorff space that contains a subset homeomorphic to the Cantor set, then Rudin has shown [same Proc. 7 (1956), 825–830; MR 18, 587] that $C(Q)$ contains a maximal closed proper subalgebra which separates the elements of Q, and which is not a maximal ideal.

P. Civin (Eugene, Ore.).

Rudin, Walter. Factorization in the group algebra of the real line. Proc. Nat. Acad. Sci. U.S.A. 43 (1957), 339–340.

Let L be the usual algebra of complex Lebesgue-integrable functions on the real line. The author shows that every member of L is the convolution of two members g and h of L, indeed such that g is in the closed ideal generated by f, and both h and its Fourier transform are positive.

R. P. Boas, Jr. (Evanston, Ill.).

Willcox, Alfred B. Note on certain group algebras. Proc. Amer. Math. Soc. 7 (1956), 874–879.

The author discusses the L_1-algebra of the direct product of a compact group and a locally compact abelian group. He proves that it is a generalized Silov algebra in the sense of his previous paper [Pacific J. Math. 6 (1956), 177–192; MR 18, 53], that it is Tauberian, and that the Ditkin theorem on closed ideals holds.

I. Kaplansky.

Hoffman, Kenneth; and Singer, I. M. Maximal subalgebras of $C(\Gamma)$. Amer. J. Math. 79 (1957), 295–305.

Let G be a discrete abelian group containing a semi-group G_+ such that $G_+\cup G_+^{-1}=G$ and G_+ is a maximal semi-group. Let Γ be the (compact) character-group of G. Each point λ in G induces a character χ_λ on Γ. By a "polynomial" we mean a finite linear combination $\sum c_\lambda\chi_\lambda$ of characters with $\lambda \in G_+$. Let $\mathfrak{A}$ be the closure on Γ under the uniform norm of the space of polynomials. Equivalently, $\mathfrak{A}$ is the space of continuous functions on Γ whose Fourier coefficients vanish at each point of G which is not in G_+. Then $\mathfrak{A}$ is a proper closed subalgebra of $C(\Gamma)$, the algebra of all continuous functions on Γ. The author's result takes its simplest form when G is a subgroup of the reals, taken in the discrete topology, and G_+ is the set of non-negative reals in G. Theorem 1: (In this case) $\mathfrak{A}$ is a maximal subalgebra of $C(\Gamma)$. In the general case, take G_0 to be the subgroup of G consisting of the elements of G_+ having their inverses in G_+. Set $\mathscr{G}=G/G_0$ and $\mathscr{G}_+=G_+/G_0$ and let Λ be the closed subgroup of Γ which consists of the characters of G which are identically 1 on G_0. Then the character group of $\mathscr{G}$ can be identified with Λ. Let $\mathfrak{A}_0$ be the subalgebra of $C(\Gamma)$ of those functions whose restrictions to Λ have Fourier transforms vanishing outside $\mathscr{G}_+$. Theorem 2: $\mathfrak{A}_0$ is a maximal subalgebra of $C(\Gamma)$ containing $\mathfrak{A}$. Further, $\mathfrak{A}=\mathfrak{A}_0$ if and only if G is archimedean linearly ordered with G_+ the set of non-negative elements (and so G is embeddable in the reals). As a special case of Theorem 1 take G to be the integers. Then Γ is the group of complex numbers of modulus one. A "polynomial" is then a finite sum $\sum_{n>0} c_ne^{in\theta}$. The algebra $\mathfrak{A}$ here consists of all continuous functions on Γ which admit analytic extensions to $|z|<1$. The maximality of this algebra was proved by the reviewer [Proc. Amer. Math. Soc. 4 (1953), 866–869; MR 15, 440] by use of special properties of the unit circle. The authors' proof, on the other hand, is based on general properties of function algebras and is very short and elegant. An essential tool in their proof is this: If A is a commutative Banach algebra and m a multiplicative linear functional on A, then there exists a real-valued measure $d\mu$ on the Silov boundary S of A such that for all f in A, $m(f)=\int_S fd\mu$. As a second special case of Theorem 1, take G to be the group of all numbers $n+m\alpha$, where n, m are integers and α is a fixed irrational. Then Γ is the torus and $\mathfrak{A}$ is the algebra of all functions on Γ with Fourier series $\sum_{n+m\alpha>0} C_{nm}e^{in\theta}e^{im\varphi}$. In the last section, the authors discuss the general problem of finding maximal subalgebras of $C(X)$, where X is a compact space.

J. Wermer (Providence, R.I.).

Korenblyum, B. I. Generalization of Wiener's Tauberian theorem and the spectrum of fast growing functions. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 280–282. (Russian)

Let $L(\alpha)$ for a given $\alpha>0$ denote the Banach algebra of measurable complex-valued functions $f(x)$, $-\infty<x<+\infty$ with the norm

$$\|f\|=\int_{-\infty}^{+\infty}|f(x)|e^{\alpha|x|}dx<+\infty$$

and the convolution as multiplication. The author announces the following results. Let $I_{\mathfrak{M}}$ be the ideal generated by a subset $\mathfrak{M}$ of $L(\alpha)$. In order that $I_{\mathfrak{M}}\neq L(\alpha)$ the following is necessary and sufficient. 1) For each value z in the strip $|\mathrm{Im}\, z|\leq\alpha$ at least one of the Fourier transforms $F(z)=\int_{-\infty}^{+\infty}f(x)e^{-ixz}dx$ of $f\in\mathfrak{M}$ be different from zero. 2) Let $\gamma^+(f)=\lim\limits_{x\to\infty}\sup\{e^{-\pi|x|/2\alpha}\log|F(x)|\}$, $\gamma^+(\mathfrak{M})=\sup\limits_{f\in\mathfrak{M}}\gamma^+(f)$ and let $\gamma^-(\mathfrak{M})$ be obtained from $\gamma^+(\mathfrak{M})$ by replacing x by $-x$. Then $\gamma^+(\mathfrak{M})=\gamma^-(\mathfrak{M})=0$. The necessity of the last

condition is explained by the existence in $L(\alpha)$ of ideals of the form $\gamma^{\pm}(\mathfrak{M})\leq c<0$. Let $M(\alpha)$ denote the conjugate to $L(\alpha)$, which consists of all functions g with $\|g\|=\sup \operatorname{ess}\{e^{-\alpha|x|}|g(x)|\}<+\infty$. If g is not equivalent to zero, then the weak closure (in the sense of the pairing $L(\alpha)$, $M(\alpha)$) of the set of all translates of g contains at least one function of one of the types $e^{-i\lambda x}$, $|Im\,\lambda|\leq\alpha$, or $e^{-i\mu x}/\Gamma(\frac{1}{2}+2\alpha xi/\pi)$ or $e^{-i\mu x}/\Gamma(\frac{1}{2}-2\alpha xi/\pi)$ with real μ. The author calls these values of λ and μ the harmonic and the non-harmonic spectrums of g. There is a description of functions g with vanishing non-harmonic spectrum and a small harmonic spectrum. Finally, the first quoted theorem is applied to give Tauberian theorems of the type: $\int_0^\infty k(t/x)d\varphi(t)\sim\int_0^\infty k(t/x)d\psi(t)$ for $x\to\infty$ implies $\varphi(x)\sim\psi(x)$ for $x\to\infty$. Here the kernel k satisfies certain regularity conditions, and φ, ψ are positive and increasing functions such that $\varphi(0)=\psi(0)=0$, $\varphi(y)/\varphi(x)\leq(y/x)^\alpha$ for $y>x$. Tauberian theorems of this type, derived by different methods, were given by M. V. Keldyš [Trudy Mat. Inst. Steklov. **38** (1951), 77–86; MR **13**, 738] and the author [Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 173–176; MR **17**, 605]. *G. G. Lorentz* (Detroit, Mich.).

Saworotnow, Parfeny P. On a generalization of the notion of H^*-algebra. Proc. Amer. Math. Soc. **8** (1957), 49–55.

Banach algebras A are considered for which the underlying Banach space is a Hilbert space. A is called right complemented if and only if the orthogonal complement of a right ideal is a right ideal; A is called right well complemented if and only if it is semi-simple and right complemented and the left annihilator of every proper right ideal is non-zero. Left complemented, complemented, left well complemented, and well complemented are defined similarly. The structure of such algebras is determined. A semi-simple right complemented A is a direct sum of simple ideals, each of which is right complemented. Every right well complemented A is a two-sided H^*-algebra and if simple is isomorphic to a canonical type of example. Every simple well complemented A is of a canonical type. *W. Ambrose* (Cambridge, Mass.).

Saworotnow, Parfeny P. On the imbedding of a right complemented algebra into Ambrose's H^*-algebra. Proc. Amer. Math. Soc. **8** (1957), 56–62.

Extending the results of the paper reviewed above, it is shown that every semi-simple right complemented A is left complemented, and, if such an A is simple, that it is isomorphic to an algebra of operators a of Hilbert-Schmidt type on a Hilbert space, for which

$$\operatorname{tr}((a\alpha)^*a\alpha)<\infty,$$

where α is some (unbounded) self adjoint operator with dense domain. *W. Ambrose* (Cambridge, Mass.).

Berberian, S. K. On the projection geometry of a finite AW^*-algebra. Trans. Amer. Math. Soc. **83** (1956), 493–509.

Let A be a finite AW^*-algebra. One knows that the maximal two-sided ideals of A are in one-one correspondence with those of the center; a new short proof is given of the fact that their intersection is 0. Moreover, for any such M, A/M is again an AW^*-algebra and a finite factor; this theorem of Wright and Yen is given a new proof here.

The main point of the present paper is to discuss the ideal I generated by the projections in M. Despite the fact that I is usually not closed, one ventures to do business with A/I. It is at least a ring with involution, and its partially ordered set of projections turns out to be an irreducible continuous geometry. A "reduction theory" for the continuous geometry of A is then given in a simple explicit way in terms of the A/I's.

Generalizing the work of von Neumann and Segal, the author has constructed the enlargement of A to a regular ring C of unbounded operators [Ann. of Math. (2) **65** (1957), 224–240; MR **18**, 914]. The projections in M generate an ideal J in C. Although C/J is a simple complete $*$-regular ring, one is surprised to learn that it is not the regular enlargement of A/M. Indeed the latter seems to be nowhere in view. What is true instead is that A/I is the bounded part of C/J. *I. Kaplansky.*

Sakai, Shôichirô. On the σ-weak topology of W^*-algebras. Proc. Japan Acad. **32** (1956), 329–332.

The author proves that, if M is a W^*-algebra and N is a C^*-algebra and φ is an isomorphism (not necessarily adjoint-preserving) of M onto N, then N is a W^*-algebra and φ is σ-weakly continuous. The proof depends on a result of Dixmier [Duke Math. J. **15** (1948), 1057–1071; MR **10**, 306] giving a characterization of adjoint Banach spaces and the following theorem of the author's: If a C^*-algebra N is the adjoint space of a Banach space F, then it is a W^*-algebra and the topology $\sigma(N, F)$ of N is the σ-weak topology [Pacific J. Math. **6** (1956), 763–773; MR **18**, 811]. *C. E. Rickart* (New Haven, Conn.).

Kadison, Richard V. Irreducible operator algebras. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 273–276.

The author proves the startling fact that if a C^*-algebra of operators acting on a Hilbert space has no nontrivial invariant closed linear manifolds, then it has no nontrivial invariant manifolds whatsoever: in fact, the algebra acts n-fold transitively on the Hilbert space. Now, let ρ be a state of the C^*-algebra $\mathfrak{A}$, and

$$\mathscr{S}=\{A \text{ in } \mathfrak{A}|\ \rho(A^*A)=0\}.$$

Corollary 1: If ρ is pure, then $\mathfrak{A}/\mathscr{S}$ is complete in the inner product $(A+\mathscr{S}, B+\mathscr{S})=\rho(B^*A)$. Corollary 2: The nullspace of ρ is $\mathscr{S}+\mathscr{S}^*$ if and only if ρ is pure. Using these results, it is shown that $\mathscr{S}$ is a maximal left ideal of $\mathfrak{A}$ when and only when ρ is pure, in which case ρ is the only state whose nullspace contains $\mathscr{S}$; and that, furthermore, any closed left ideal of $\mathfrak{A}$ is the intersection of the maximal left ideals containing it. *J. Feldman* (Berkeley, Calif.).

Vulih, B. Z. Application of the theory of partially ordered spaces to the study of self-adjoint operators in Hilbert space. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 169–172. (Russian)

Expository article concerning what the title describes. Principal references are to papers of Kadison [MR **13**, 47], Sobolev [MR **14**, 385], Lyubovin [MR **18**, 660], Stone [MR **14**, 565], and Fell and Kelley [MR **14**, 480]. *B. Gelbaum* (Minneapolis, Minn.).

Štraus, A. V. Generalized resolvents of symmetric operators and eigenfunction expansions of a class of boundary problems. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 251–253. (Russian)

A formula for the generalised resolvent of a Hermitian operator A is given: $R_\lambda=(A_{F(\lambda)}-E)^{-1}$, where $F(\lambda)$ is any operator from the defect space N_λ of A to $N_{\bar\lambda}$, whose norm is not greater than 1, and which is a regular function of λ in the halfplane, and $A_{F(\lambda)}$ is an extension of A defined

by means of $F(\lambda)$. Applications to the spectral theory of differential equations are given. Proofs and details have been given by the author previously [Izv. Akad. Nauk SSSR. Ser. Mat. 18 (1954), 51–86; 19 (1955), 201–220; MR 16, 48; 17, 620]. *J. L. B. Cooper* (Cardiff).

Brodskiĭ, M. S. On Jordan cells of infinite-dimensional operators. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 926–929. (Russian)

The author says that a linear operator A in a Hilbert space H is unicellular if the invariant subspaces of A are totally ordered with respect to inclusion. Suppose that A has the following properties: the operator $i^{-1}(A-A^*)$ has finite rank r, with eigenvalues $\omega_\alpha>0$ and orthonormal eigenvectors e_α $(1\leqq\alpha\leqq r)$; the spectrum of A consists of the point 0 only; and the system of vectors

$$\{A^n e_\alpha:\ n\geqq 0,\ 1\leqq\alpha\leqq r\}$$

is fundamental in H. Then A has a characteristic matrix-function $W(\lambda)$ [M. S. Brodskiĭ, same Dokl. (N.S.) 97 (1954), 761–764; MR 16, 836], which satisfies the inequality $\|W(\lambda)\|\leqq e^{l/|\lambda|}$, where $l=\sum_{\alpha=1}^r \omega_\alpha$. A sufficient condition for A to be unicellular is that this inequality be exact in the sense that, given $\varepsilon>0$, there is a sequence $\lambda_k\to 0$ for which

$$\|W(\lambda_k)\|>e^{(l-\varepsilon)/|\lambda_k|}(k\geqq 1).$$

As an application, the author considers the operator A defined in $L^2(0, l)$ by the equation

$$Af(x)=i\int_0^x \sum_{\alpha=1}^r \varphi_\alpha(x)\overline{\varphi_\alpha(t)}f(t)\,dt,$$

where $\sum_{\alpha=1}^r |\varphi_\alpha(x)|^2$ is different from 0 almost everywhere on $(0, l)$. The following results are obtained. If each $\varphi_\alpha(x)$ has an absolutely continuous derivative, then A is unicellular; the sequence $(A^n f_0)$ is then dense in $L^2(0, l)$ if and only if the set

$$\{x:\ 0\leqq x\leqq\varepsilon,\ f_0(x)\neq 0\}$$

has positive measure for all $\varepsilon>0$. On the other hand, if $0=x_0<x_1<\cdots<x_p=l$ is a subdivision of $(0, l)$ and the vector

$$\xi(x)=[\varphi_1(x), \varphi_2(x), \cdots, \varphi_r(x)]$$

takes the constant value ξ_k in the interval (x_{k-1}, x_k) $(1\leqq k\leqq p)$, then A is unicellular if and only if no two successive vectors ξ_k are orthogonal. *F. Smithies.*

Harazov, D. F. On the spectral theory of completely continuous operators. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 544–546. (Russian)

The author considers, in a Hilbert space X, equations of the form

$$(E-\lambda A_1-\lambda^2 A_2)x=y.$$

where E is the identical operator, A_1 and A_2 are linear and completely continuous (compact), and there is a positive definite bounded self-adjoint operator H such that

$$(HA_i x, y)=(x, HA_i y)\ (i=1, 2),\ (HA_2 x, x)\geqq 0,$$

for all $x, y\in X$. The result of some earlier papers [same Dokl. (N.S.) 91 (1953), 1023–1026; 102 (1955), 693–696; Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 20 (1954), 297–315; MR 15, 881; 17, 1114, 16, 934] are extended to this more general case. No proofs are given. *F. Smithies* (Cambridge, England).

Steinberg, R. Note on a theorem of Hadwiger. Pacific J. Math. 6 (1956), 775–777.

Let H be a Hilbert space over the real or complex numbers. This paper consists chiefly of conclusions which can be drawn from the existence of a Parseval relation in H. If $\{u_\alpha\}$ is an indexed set of elements of H, assume for each $(x, y)\in H$ that $(x, y)=\sum_\alpha (x, u_\alpha)(u_\alpha, y)$. The theorem of Hadwiger [Comment. Math. Helv. 13 (1940), 90–107; MR 2, 260] states that if the Parseval relation holds for $\{u_\alpha\}$, then $\{u_\alpha\}$ is the projection of an orthonormal basis in a Hilbert space $K\supset H$. The author gives a simple proof of this theorem and of other theorems giving conditions implying similar conclusions for sets $\{u_\alpha\}$ in H.
R. E. Fullerton (College Park, Md.).

Vaĭnberg, M. M. Certain questions of functional analysis and variational methods of investigation of non-linear equations. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 1(73), 162–165. (Russian)

This is a short account of the application of critical point methods to solution of nonlinear operational equations, particularly to potential operators. Proofs have been given in a previous article by the author [Uspehi Mat. Nauk (N.S.) 7 (1952), no. 4(50), 55–102; MR 14, 384].
J. L. B. Cooper (Cardiff).

See also: Zuhovickiĭ, p. 30; Zuhovickiĭ and Stečkin, p. 30; Davis and Fan, p. 30; Karcivadze, p. 42; Trjitzinsky, p. 50.

TOPOLOGY

General Topology

Ford, L. R. Interval-additive propositions. Amer. Math. Monthly 64 (1957), 106–108.

A statement P concerning intervals is interval-additive if whenever P is true for each of two overlapping intervals (with common interior points) it is also true for the interval obtained by combining them. For example, the proposition that an interval is more than one unit long is interval-additive; that an interval is less than one unit long is not.

Fundamental Theorem: If P is an interval-additive proposition which is true at each point of a closed interval $a\leqq x\leqq b$, i.e., for some subinterval containing the point, then P is true in the whole interval. This theorem is used to give a unified treatment of several results concerning a real function $f(x)$ such as: if $f(x)$ is continuous in $a\leqq x\leqq b$ it is bounded; a continuous function assumes its supremum and its infimum; if $f(x)$ is continuous on $a\leqq x\leqq b$ it is uniformly continuous. A corollary of the main theorem is: If P is an interval-additive proposition which is not true for a closed interval I, then there is at least one point ξ of I such that the proposition is not true for any subinterval of I which contains ξ. Since the statement that an interval contains a finite number of points of a set S is interval-additive, the corollary gives immediately the Bolzano-Weierstrass theorem.
R. L. Jeffery (Kingston, Ont.).

Bruns, Günter. Durchschnittsdarstellungen von Filtern. Math. Ann. **133** (1957), 26–38.

This paper is the first of several on the structure of filters. It contains the elementary analysis and is mainly concerned with the representation of a filter as the intersection of two others. The complete distributivity of the lattice of all filters on a set is important to the argument. Conditions are given under which, if G, F are filters, and $G \supset F$, there is a filter H such that $F = G \cap H$. Necessary and sufficient conditions are given in order that one of the factors G or H be a principal filter.

M. E. Shanks.

Bernstein, I. Applications quasi-monotones et revêtements. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 117–121, XI–XII, (Russian summary)

If X and Y are connected, locally connected spaces that have universal coverings, and ϕ is a continuous map of X on Y such that, for some point $y \in Y$, $\phi^{-1}(y)$ is either compact or has a finite number of components, then $\phi(\pi(X))$ is of finite index in $\pi(Y)$. This generalizes a similar result of Eilenberg and Whyburn about the 1-dimensional homology group and has a slight generalization in which X and Y are not required to have universal covering spaces. An example is given of a unicoherent 3-dimensional manifold doubly covered by $S^2 \times S^1$, which is multicoherent.

R. H. Fox.

Morita, Kiiti. On closed mappings. Proc. Japan Acad. 32 (1956), 539–543.

The following are the principal results. A) Let Y be the image, under a closed continuous mapping f, of a paracompact, locally compact space. Then Y is paracompact; moreover, Y is locally compact if and only if the boundary of $f^{-1}(y)$ is compact for every y in Y. B) The image of a paracompact and perfectly normal space, under a closed, continuous mapping, is paracompact and perfectly normal. C) Let f be a closed, continuous mapping from a paracompact, locally compact space X onto a locally compact space Y. Then f can be extended to a continuous mapping from $\gamma(X)$ onto $\gamma(Y)$, where $\gamma(X)$ and $\gamma(Y)$ are the Freudenthal compactifications of X and Y respectively. {Reviewer's note: Both B) and the first part of A) follow from the following recent result of the reviewer [Bull. Amer. Math. Soc. **62** (1956), 414]: The image of a paracompact space, under a closed, continuous mapping, is paracompact.}

E. Michael (Seattle, Wash.).

Morita, Kiiti. On decomposition spaces of locally compact spaces. Proc. Japan Acad. **32** (1956), 544–548.

All spaces in this review are Hausdorff. Y is a decomposition space of X if there exists a map f from X onto Y such that Y has the finest topology which makes f continuous. After obtaining a necessary and sufficient condition for a space Y to be a decomposition space of a locally compact Lindelöf (respectively paracompact) space, the author obtains Theorem 1: If Y is a decomposition space of a locally compact Lindelöf space X, then Y is paracompact. {Reviewer's note: Since Y is clearly Lindelöf, it would be sufficient to show that Y is regular, but there seems to be no simple way to do this directly.} Another result is Theorem 2: Y is the image of a locally compact, paracompact (respectively locally compact, metrizable) space under an open, continuous mapping if and only if Y is locally compact (respectively locally compact and locally metrizable).

E. Michael.

Morita, Kiiti. Note on mapping spaces. Proc. Japan Acad. 32 (1956), 671–675.

In the paper reviewed above, the author has introduced the notion of a Hausdorff space having the weak topology with respect to compact sets in the wider sense. A space X has this property if, when a subset A of X is such that $A \cap K$ is closed for every compact subset K of X, then A is closed. Examples of spaces having this property are locally compact spaces, CW-complexes, and spaces satisfying the first axiom of countability. The author proves the following generalization of a classical theorem. If $X \times Y$ is a Hausdorff space having the weak topology with respect to compact sets in the wider sense, then the natural map of functions spaces θ: $Z^{X\times Y} \to (Z^X)^Y$ is a homeomorphism.

J. C. Moore (Princeton, N.J.).

McAuley, Louis F. A note on naturally ordered sets in semi-metric spaces. Proc. Amer. Math. Soc. **8** (1957), 384–386.

The note shows that if (1) G is any naturally ordered collection of pairwise disjoint subsets of a semimetric space with continuous distance function, and (2) the set-sum of the elements of G is hereditarily separable, then at most a countable number of the elements g of G can contain a point that is not a condensation point both of the collection P_g of all predecessors of g in G, and the collection F_g of all successors of g in G.

For separable metric spaces, the result goes back to Zarankiewicz [Fund. Math. **12** (1928), 121–125].

L. M. Blumenthal (Columbia, Mo.).

Wagner, K. Über infinitesimale Kerne von Punktmengen in topologischen Räumen. Math. Ann. **133** (1957), 52–78.

In an earlier paper [K. Dörge and K. Wagner, Math. Ann. **123** (1951), 1–33; MR **13**, 635] the concept of (infinitesimal) nucleus of a set M at a limit point p was developed. The nucleus is the intersection of a family V^* of sets (e.g., half-cones with vertex p). This paper begins by axiomatizing these V^* families, and defining derivatives, both strong and weak, and relating these to the nucleus. Several results present upper bounds for the dimension of M in terms of the dimension of all (or almost all) nuclei. Further, there are results on the countability of those points of a continuum in R^n, where several derivatives exist. The emphasis is throughout on familiar spaces; there are many examples and references.

R. Arens.

Mohat, John T. Concerning spirals in the plane. Duke Math. J. **24** (1957), 249–264.

In answer to a question raised earlier by R. L. Moore, the author exhibits the existence of an equicontinuous collection of disjoint arcs filling up the interior of a rectangle, each arc of which has one end point on the lower base and one on the upper base of the rectangle and spirals down on just one point. Indeed it is shown that if M is any 0-dimensional inner limiting set interior to the rectangle whose projection on the base fills up the interior of the base and such that no vertical line contains two points of M, then for any rectangle R there exists an equicontinuous collection of arcs of the above sort filling up the interior of R and such that the set M' of all points x such that some arc of the collection spirals down on x is homeomorphic with M.

G. T. Whyburn.

Trjitzinsky, W. J. Aspects topologiques de la théorie des fonctions réelles et quelques conséquences dynamiques. Ann. Mat. Pura Appl. (4) 42 (1956), 51–117.

This paper is chiefly concerned with extensions of results of Denjoy [Leçons sur le calcul des coefficients d'une série trigonométrique, Parties I–IV, Gauthier-Villars, Paris, 1941, 1949; MR **8**, 260; **11**, 99] and the author [Acta Math. **95** (1956), 191–289; MR **17**, 1247] to the case where the underlying space X is separable metric (Hilbert space is used as an illustration, but no significant use is made of its additional properties.) The first part develops convergence properties of sequences of continuous functions and of families of such functions. The following is an example of one of the numerous theorems proved. Let P be a perfect subset of Hilbert space, let T be a set in a euclidean space, let t_0 be an accumulation point of T, $t_0 \notin T$, let f be a continuous real-valued function defined on $P\times T$, and for $x \in P$ let $h(x)$ be the set of limit points of $f(x, t)$ as t converges to t_0. Let λ be a real number or $+\infty$ or $-\infty$, and suppose that $\lambda \in h(x)$ for each $x \in P$. Then to every pair α, β of positive real numbers there corresponds a sequence $t_1, t_2, \cdots$ in T such that $|t_0-t_j|<\beta$ $(j=1, 2, \cdots)$, and 1) $|f(x, t_{j(x)})-\lambda|<\alpha$ if λ is finite; 2) $f(x, t_{j(x)})>\alpha^{-1}$ if $\lambda=+\infty$; 3) $f(x, t_{j(x)})<-\alpha^{-1}$ if $\lambda=-\infty$.

The second part deals with convergence properties of families of sets (chiefly one-parameter). Again an example: For each t, $t_0\leqq t<+\infty$, let $G(t)$ be a closed subset of X and let $L_f{}^+$ be the set of all points q with the property that there exists for each $t\geqq t_0$ a point $N(q, t) \in G(t)$ such that $\lim_{t\to\infty} N(q, t)=q$. Then a necessary and sufficient condition that the closed set E be contained in $L_f{}^+$ is that for each $\delta>0$ there exist a decomposition $E=\bigcup_{i=1}^{\infty} E_i$ of E into subsets, each of diameter less than δ, such that $\lim\sup_t h_i(t)\leqq\delta$ $(i=1, 2, \cdots)$, where $h_i(t)=\max_{q\in E_i} d(q, G(t))$. There are applications of these results to topological stability properties of sets in a continuous flow. *G. A. Hedlund* (New Haven, Conn.).

See also: Pinsker, p. 7; Sucheston, p. 21; Rudin, p. 46; Fet, p. 53; Kolmogorov, p. 69.

Algebraic Topology

Yamanoshita, Tsuneyo. On certain cohomological operations. J. Math. Soc. Japan 8 (1956), 300–344.

L'auteur étudie d'une part les "opérations de Bockstein" d'ordre supérieur, d'autre part une généralisation du carré de Pontrjagin. X désigne un espace quelconque. Soit p un entier $\geqq 2$, fixé et provisoirement quelconque. Pour chaque entier $r\geqq 1$, définissons des sous-groupes $D_r{}^n\subset C_r{}^n\subset H^n(X, Z_p)$ comme suit: $\xi \in C_r{}^n$ s'il existe dans la classe ξ un cocyle x tel que δx ait la forme $p^r y$; $\xi \in D_r{}^n$ s'il existe dans la classe ξ un cocyle x tel que $p^{r-1}x$ ait la forme δz. On a

$$D_1{}^n=0,\ C_1{}^n=H^n(X; Z_p),\ D_r{}^n\subset D_{r+1}{}^n\subset C_{r+1}{}^n\subset C_r{}^n.$$

Dans $H_r{}^n(X; Z_p)=C_r{}^n/D_r{}^n$ on a un endomorphisme Δ_r de degré $+1$, qui à la classe d'un x tel que $\delta x=p^r y$ associe la classe de y; Δ_1 est l'opérateur de Bockstein $H^n(X; Z_p)\to H^{n+1}(X; Z_p)$. Le noyau de Δ_r est l'image de $C_{r+1}{}^n$ dans $H_r{}^n$, l'image de Δ_r est l'image de $D_{r+1}{}^n$ dans $H_r{}^n$, donc le groupe de Δ_r-cohomologie de $H_r{}^n$ s'identifie à $H_{r+1}{}^n$. Si un $\alpha \in H_r{}^n$ est tel que sa classe dans $H_r{}^n/D_s{}^n$ $(s>r)$ soit dans $H_s{}^n$, on dit par abus de langage que $\alpha \in H_s{}^n$.

Chaque $H_r{}^n$ est un foncteur contravariant de l'espace X. De plus la suite exacte de cohomologie (pour $Y\subset X$):

$$\cdots\to H^n(X, Y; Z_p)\xrightarrow{j^*} H^n(X; Z_p)\xrightarrow{i^*} H^n(Y; Z_p)\xrightarrow{\delta^*} H^{n+1}(X, Y; Z_p)\to\cdots$$

induit une suite (non exacte en général)

$$\cdots\to H_r{}^n(X, Y; Z_p)\to H_r{}^n(X; Z_p)\to H_r{}^n(Y; Z_p)\to H_r{}^{n+1}(X, Z; Y_p)\to\cdots$$

L'auteur étudie diverses relations entre i^*, j^*, δ^* et les Δ_r (§ 3 de l'article).

Supposons maintenant p premier, et soit $v \in H_r{}^n(X; Z_p)$; si n est impair, on a évidemment $v^p=0$ sauf si $p=2$, et alors v^2 est une classe entière, donc $\Delta_s(v^2)=0$ pour tout s. Si n est pair, on a $v^p \in H_{r+1}{}^{np}(X; Z_p)$; l'auteur démontre que $\Delta_{r+1}(v^p)=v^{p-1}\cdot(\Delta_r v)$, sauf si $p=2$ et $r=1$, auquel cas $\Delta_2(v^2)=v\cdot(\Delta_1 v)+\mathrm{Sq}^n(\Delta_1 v)$. La démonstration fait intervenir les applications D_i de Steenrod, et a un lien étroit avec les "puissances de Pontrjagin": ce sont des opérations cohomologiques

$$'P_p{}^r: H^n(X; Z_{p^r})\to H^{np}(X; Z_{p^{r+1}})$$

définies et étudiées au § 4 [l'auteur n'avait pas eu connaissance de la Note de Thomas, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 266–269; MR **18**, 57]. L'opération $'P_p{}^r$ jouit des propriétés suivantes: l'application composée

$$H^n(X; Z_{p^r})\xrightarrow{'P_p{}^r} H^{np}(X; Z_{p^{r+1}})\xrightarrow{k} H^{np}(X; Z_{p^r}),$$

où k est induit par la projection naturelle $Z_{p^{r+1}}\to Z_{p^r}$, est la puissance p-ième. De plus

$$'P_p{}^r(u+v)-{}'P_p{}^r(u)-{}'P_p{}^r(v)=f_p\left(\sum_{j=1}^{p-1}\frac{1}{p}\binom{p}{j}u^j v^{p-j}\right),$$

où f_p désigne l'homomorphisme

$$H^{np}(X; Z_{p^r})\to H^{np}(X; Z_{p^{r+1}})$$

qu'induit la multiplication des cochaînes par l'entier p. L'auteur étudie surtout une opération $P_p{}^r$ qui applique un sous-groupe de $H^n(X; Z_p)$ dans un quotient de $H^{np}(X; Z_{p^{r+1}})$, mais que le rapporteur juge moins intéressante puisqu'elle se déduit évidemment de $'P_p{}^r$.

Le texte comporte beaucoup de calculs, et des diagrammes inutilement compliqués (on compte souvent plus de 70 flèches par diagramme); il n'est pas possible de signaler toutes les erreurs d'impression. *H. Cartan.*

★ **Adem, José. The relations on Steenrod powers of cohomology classes. Algebraic geometry and topology.** A symposium in honor of S. Lefschetz, pp. 191–238. Princeton University Press, Princeton, N. J., 1957. $7.50.

L'auteur donne des démonstrations détaillées des relations entre opérations de Steenrod itérées qu'il avait annoncées antérieurement [Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 720–726; **39** (1953), 636–638; MR **14**, 306; **15**, 53]. Dans l'intervalle a paru la démonstration de H. Cartan [Comment. Math. Helv. **29** (1955), 40–58; MR **16**, 847] qui utilise une tout autre méthode. L'A. adopte el point de vue de Steenrod [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 213–217, 217–223; MR **14**, 1005, 1006]: soient m et n deux entiers >1; soit S_n le groupe symétrique de degré n; soit Z_m (resp. $\bar{Z}_m$) le groupe additif des entiers mod. m, considéré comme S_n-module dans lequel S_n opère trivialement (resp. dans lequel chaque $\alpha \in S_n$ multiplie chaque élément de Z_m par la signature de α). Pour chaque entier q pair (resp. impair) chaque élément

$c \in H_t(S_n; Z_m)$ (resp. $c \in H_t(S_n; \hat{Z}_m)$) définit une opération cohomologique $H^q(X; Z_m) \to H^{nq-t}(X; Z_m)$, appelée n-ième puissance réduite suivant c. Quand n et m sont égaux à un même p premier, on obtient ainsi les classiques opérations de Steenrod.

L'A. étudie le cas où $n=p^2$ et $m=p$; il détermine un système de générateurs des $H_t(S_{p^2}; Z_p)$ pour cela, il considère un p-groupe de Sylow G de S_{p^2} et utilise le fait (connu grâce à la théorie du transfert) que l'homomorphisme f_t: $H_t(G; Z_p) \to H_t(S_{p^2}; Z_p)$ est surjectif. Il détermine explicitement les $H_t(G; Z_p)$ et constate que toute opération de Steenrod itérée deux fois peut être définie par un élément de $H(G; Z_p)$. Pour trouver des relations entre ces opérations itérées, il suffit de connaître des éléments du noyau N_t de f_t. (On évite de considérer les coefficients dans $\hat{Z}_p$ parce que toute relation entre opérations de Steenrod itérées qui est valable sur les degrés q pairs l'est universellement). Or si $x \in H_t(G; Z_p)$ et si ω est un automorphisme de G induit par un automorphisme intérieur de S_{p^2}, on a $x-\omega x \in N_t$. L'auteur obtient ainsi les relations désirées entre opérations de Steenrod itérées. Cette méthode ne prouve pas que toute autre relation soit conséquence de celles-là; mais il en est ainsi (méthode de H. Cartan), ce qui montre inversement que l'auteur a obtenu ici toutes les relations existant entre les générateurs de $H(S_{p^2}; Z_p)$.

Les calculs sont délicats, et conduits avec grand soin; d'une manière générale, l'exposition de ce sujet difficile est faite avec beaucoup de clarté et de précision. Comme application, on montre que toute opération $\wp_p{}^i$ de Steenrod s'exprime (dans l'algèbre des opérations cohomologiques) comme polynôme par rapport aux $\wp_p{}^k$ telles que k soit une puissance de p. Les applications topologiques de la théorie sont promises pour un travail ultérieur.

H. Cartan (Paris).

Adem, José. A cohomology criterion for determining essential compositions of mappings. Bol. Soc. Mat. Mexicana (2) **1** (1956), 38–48. (Spanish)

Pour chaque application continue f d'un espace X en un espace Y et chaque opération cohomologique stable [collection d'homomorphismes θ^α: $H^q(K, L) \to H^{q+\alpha}(K, L)$ définis pour toute paire K, L et tout entier q; le groupe de coefficients ne sera pas explicité], l'A. introduit l'opération cohomologique fonctionnelle $\theta_f{}^\alpha$ suivant une méthode due à Steenrod [Ann. of Math. (2) **50** (1949), 954–988; MR **11**, 122]. $\theta_f{}^\alpha$ est définie dans un certain sous-groupe qu'elle applique dans un groupe-quotient.

On a $\theta_f{}^\alpha=0$ lorsque f est non-essentielle. Ayant des applications g de W en X et f de X en Y, on peut cependant avoir (en désignant par $'\theta_g{}^\beta$ l'opération induite par $\theta_g{}^\beta$ par passage au quotient) $'\theta_g{}^\beta\theta_f{}^\alpha u \neq 0$ pour un $u \in H^*(Y)$ même si l'application composée fg est non-essentielle (parce que $'\theta_g{}^\beta\theta_f{}^\alpha$ n'est pas $\theta_{(fg)}$. L'A. donne le critère suivant pour que l'existence d'un $u \in H^n(Y)$ tel que $'\theta_g{}^\beta\theta_f{}^\alpha u \neq 0$ entraîne que fg est essentielle, lorsque $H^{n-1}(W)=H^{n+\alpha+\beta-1}(Y)=0$. Appelons "mapping cone" (par analogie avec le "mapping cylinder") de l'application f de X en Y l'espace déduit de $CX \cup Y$, où dans le cone CX [obtenu à partir de $X \times E^1$, voir Spanier et Whitehead, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 655–660; MR **15**, 52] chaque point $x \times 1$ de $X \times 1$ est identifié avec $f(x) \in Y$. Soit SW la suspension de W. Si, pour toute application h de SW dans le mapping cone de f et pour tout u'' (élément du n^e groupe de cohomologie du mapping cone de h) qui est appliqué (par l'inclusion de Y dans le mapping cone) sur u, on a $\theta^\beta\theta^\alpha u''=0$, alors fg est essentielle. [La démonstration repose sur la possibilité de factoriser la suspension de g lorsque fg est non-essentielle: SW est appliqué par une application h sur le mapping cone de f qui est appliqué (par identification de Y en un point) sur SX; pour cette h, on ne pourrait avoir $\theta^\beta\theta^\alpha u''=0$ si $'\theta_g{}^\beta\theta_f{}^\alpha u \neq 0$.]

L'A. applique ce résultat à la composition d'applications g: $S^m \to S^k$, f: $S^k \to S^n$ où u est un générateur de $H^n(S^n)$, $\wp_g{}^\beta\wp_f{}^\alpha u \neq 0$, $\alpha \geqq \beta$ ($\wp$ désignant les p-puissances de Steenrod); fg et sa suspension itérée r fois sont essentielles dans les cas suivants:

$$p=2,\ \alpha=\beta,\ 0 \leqq r < \infty,$$
$$p=2,\ \alpha>\beta,\ 0 \leqq r < \alpha+\beta-n,$$
$$p=3,\ \alpha=\beta,\ 0 \leqq r < 4\alpha-n,$$
$$p=3,\ \alpha>\beta,\ 0 \leqq r \leqq 2\alpha+2\beta-n,$$
$$p>3,\ \alpha \geqq \beta,\ 0 \leqq r \leqq 2\alpha+2\beta-n.$$

Par un autre exemple, l'A. retrouve la p-composante de $\pi_{4p+n-5}(S^n)$, $n \geqq 3$.

G. Hirsch (Bruxelles).

Saito, Yoshihiro. On the homotopy groups of Stiefel manifolds. J. Inst. Polytech. Osaka City Univ. Ser. A. **6** (1955), 39–45.

In this paper the author determines additional results on the homotopy of the Stiefel manifolds of m-frames in Euclidean n-space. His work is an extension of work of J. H. C. Whitehead [Proc. London Math. Soc. (2) **48** (1944), 243–291; MR **6**, 279], and of Barratt and Paechter [Proc. Nat. Acad. Sci. U.S.A. **38** (1951), 119–121; MR **13**, 674]. The method used is to reduce the problem of computing the groups in question to that of computing the homotopy groups of the space obtained by taking a j-dimensional real projective space and shrinking a $(k-1)$-dimensional hyperplane to a point, using a theorem of J. H. C. Whitehead, and then to compute the relevant homotopy groups of the space so constructed.

J. C. Moore (Princeton, N.J.).

Wada, Hidekazu. On the space of mappings of a sphere on itself. Ann. of Math. (2) **64** (1956), 420–435.

Various authors have made efforts to determine when $\pi_{2n+1}(S^{n+1})$ contains an element of Hopf invariant 1. Among others, G. W. Whitehead [Ann. of Math. (2) **51** (1950), 192–237; MR **12**, 847] proved that $\pi_{2n+1}(S^{n+1})$ contains an element of Hopf invariant 1 if and only if the Whitehead product of the generator of $\pi_n(S^n)$ vanishes. The author of this paper obtains an essentially equivalent condition with that of Whitehead. Let $G_n{}^*$ be the space of mappings of an n-sphere S^n onto itself, and let $F_n{}^*$ be the subspace of $G_n{}^*$ whose elements keep fixed the north pole of S^n. Then $G_n{}^*$ and $S^n \times F_n{}^*$ have the same homotopy type if and only if $\pi_{2n+1}(S^{n+1})$ contains an element of Hopf invariant 1.

H. Uehara (Bogota).

Spanier, E. H. Duality and S-theory. Bull. Amer. Math. Soc. **62** (1956), 194–203.

Let $[X, Y]$ denote the set of homotopy classes of maps of a space X into a space Y and let S be the suspension operator. Maps f: $S^kX \to S^kY$, g: $S^lX \to S^lY$ belong to the same S-class if there exist integers p, q such that $k+p=l+q$ and S^pf is homotopic to S^qg. S-classes are called S-maps and $\{X, Y\}$ denotes the set of S-maps from X to Y. Then $\{X, Y\}$ may be given the structure of an abelian group, and if Y is $(n-1)$-connected and X is a complex of dimension $\leqq 2n-2$, then there is a (1-1)

correspondence between $[X, Y]$ and $\{X, Y\}$. Obviously $\{X, Y\}=\{SX, SY\}$.

The present paper is a summarized account of the S-theory developed by the author and J. H. C. Whitehead [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 655–660; Mathematika **2** (1955), 56–80; Algebraic geometry and topology, Princeton, 1957, pp. 330–360; MR **15**, 52; **17**, 653; **18**, 919]. The most striking feature is a rigorous duality theory comprehending the formal analogy between homotopy and cohomotopy. Within this theory, for example, the Hurewicz isomorphism theorem and the Hopf classification theorem appear as consequences of each other.

P. J. Hilton (Manchester).

Adams, J. F. Four applications of the self-obstruction invariants. J. London Math. Soc. **31** (1956), 148–159.

This paper starts with an exposition of certain "self-obstruction" invariants which although formulated in a different fashion, are the well-known k invariants of Postnikov. These invariants are formulated in a form which is convenient in the study of the homotopy type of CW-complexes. After this the author characterizes the so-called J_m complexes introduced by J. H. C. Whitehead [Bull. Amer. Math. Soc. **55** (1949), 213–245; MR **11**, 48]. Indeed he proves the following theorem. A simply connected complex K is a J_m complex if and only if 1) its Postnikov invariants are zero in dimension less than $m+1$, 2) $H_r(\pi_n, n; Z)=0$ for $n+1\leqq r\leqq m+1$, and 3) $\mathrm{Tor}(\pi_p, \pi_q)=0$ for $p\neq q$, $p+q\leqq m$. In this theorem π_r denotes the rth homotopy group of K.

In addition to the preceding, the paper also contains other applications of the Postnikov invariants, partiularly to problems involving the n-type of CW-complexes.

J. C. Moore (Princeton, N.J.).

Murasugi, Kunio. On the homotopy type of a CW-complex. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1955), 99–110.

In this paper the author introduces cohomology invariants to describe the homotopy type of a CW-complex. He writes: "In general, it seems to me that $l^q(X)$ and $l_r(X)$ are the geometrical realizations of Eilenberg-Maclane invariant and Postnikov invariant." In fact, the author's definitions are essentially equivalent to those adopted by J. H. C. Whitehead [Proc. London Math. Soc. (3) **3** (1953), 385–416; MR **15**, 734] and J. F. Adams [see the paper reviewed above] to describe the Postnikov invariants.

P. J. Hilton (Manchester).

Barratt, M. G.; and Whitehead, J. H. C. The first non-vanishing group of an $(n+1)$-ad. Proc. London Math. Soc. (3) **6** (1956), 417–439.

Let X be a CW-complex and $X_1, \cdots, X_n$ be sub-complexes such that $(X; X_1, \cdots, X_n)$ is an $(n+1)$-ad (i.e., the intersection of all the sub-complexes X_i is non-vacuous) which satisfies the following condition: Every point of X belongs to at least $n-1$ of the subcomplexes X_i. This paper is concerned with the determination of the first non-vanishing homotopy group of such an $(n+1)$-ad.

Let C denote the intersection $X_1\cap X_2\cap\cdots\cap X_n$ and let $A_j=X_1\cap\cdots X_{j-1}\cap X_{j+1}\cap\cdots\cap X_n$ for $1\leqq j\leqq n$. H. Toda has proven that if each pair (A_i, C) is q_i-connected, $1\leqq i\leqq n$, $q_i\geqq 2$, and C is simply connected, then the $(n+1)$-ad $(X; X_1, \cdots, X_n)$ is q-connected, where $q=q_1+q_2+\cdots+q_n$ [Proc. Japan Acad. **29** (1953), 299–304; MR **15**, 732]. The main theorem of the present paper asserts that under these hypotheses the homotopy group $\pi_{q+1}(X; X_1, \cdots, X_n)$ is isomorphic to the direct sum of $(n-1)!$ copies of $\pi_{q_1+1}(A_1, C)\otimes\cdots\otimes\pi_{q_n+1}(A_n, C)$. To define this isomorphism, the authors introduce a generalized Whitehead product. {Reviewer's note: The structure of the group $\pi_{q+1}(X; X_1, \cdots, X_n)$ has been determined independently by H. Toda in a recent paper [J. Inst. Polytech. Osaka City Univ. Ser. A. **6** (1955), 101–120; MR **17**, 773]. Although Toda's result is more general in some respect, he does not introduce the generalized Whitehead products and hence does not give explicit formulas for the isomorphism in question.}

The authors also study under what conditions the homomorphism $\pi_m(X_1, C)\otimes\pi_n(X_2,C)\rightarrow\pi_{m+n-1}(X; X_1,X_2)$ (defined by generalized Whitehead products) is an isomorphism onto, or merely onto, when $m+n-1$ is greater than the dimension of the first non-vanishing homotopy group of the triad $(X; X_1, X_2)$. For such a triad, they obtain a certain exact sequence involving the homotopy groups of (X, X_2), (X_1, C), X_2, and C, and apply this exact sequence to the study of the homotopy groups of the pair (X, X_2) in the case where X is obtained from X_2 by the adjunction of cells.

W. S. Massey.

Eilenberg, Samuel; and Ganea, Tudor. On the Lusternik-Schnirelmann category of abstract groups. Ann. of Math. (2) **65** (1957), 517–518.

This paper is a statement of results relating the dimension, category, and geometric dimension of a group. If Π is a group, then Π is of dimension n if the cohomology group $H^q(\Pi; A)=0$ for $q>n$, where A is any Π module, and n is the least such integer.

Let $K(\Pi)$ be a connected CW-complex such that $\pi_1(K(\Pi))=\Pi$, and $\pi_q(K(\Pi))=0$ for $q>i$, i.e. $K(\Pi)$ is a connected aspherical CW-complex with Π as fundamental group. The homotopy type of $K(\Pi)$ is an invariant of the group Π. The category of Π is defined to be the category of the complex $K(\Pi)$. The authors' definition of category differs from the usual one. They say that X is of category n, if n is the least integer such that X may be covered by open sets $U_0, \cdots, U_n$ such that each U_i is contractible in X. If no such integer exists, then the category of X is ∞. Note that this is just the usual definition of category $n+1$.

Finally, the authors define the geometric dimension of a group Π as the least integer n for which there is a connected aspherical CW-complex $K(\Pi)$ with Π as fundamental group which is of dimension n. Otherwise the geometric dimension of Π is ∞.

The main theorem of the paper is as follows. For any group Π

$$\dim \Pi=\text{category } \Pi=\text{geom. dim } \Pi$$

except possibly for the following three cases:

	A	B	C
dim	1	1	2
category	2	2	2
geom. dim	2	3	3

J. C. Moore (Princeton, N.J.).

★ **Hopf, H. Die Coinzidenz-Cozyklen und eine Formel aus der Fasertheorie. Algebraic geometry and topology.** A symposium in honor of S. Lefschetz, pp. 263–279. Princeton University Press, Princeton, N. J., 1957. $7.50.

Let X be a finite polyhedron, triangulated dy K, and let Y be an n-dimensional closed orientable manifold.

Given two maps $F, G:X\to Y$, Lefschetz defined the class of coincidence cocyles $\Omega(F, G)\in H^n(X)$. The author generalises this to fibre theory. Let $p:R\to X$ be a fibre bundle, base X and fibre Y (with X and Y as above), and suppose $\pi_1(X)$ acts trivially on $H^n(Y)$. Let f, g be two cross-sections over a subset of X containing the n-skeleton K^n of K. A coincidence class $\bar\omega(f, g)\in H^n(X)$ is defined, which remains unchanged if f, g are varied homotopically as cross-sections. This is done by separating f and g on K^{n-1} and measuring their intersection number over each n-simplex of K. In the special case that $R=X\times Y$, $f=1\times F$, $g=1\times G$, $\bar\omega$ reduces to Ω above.

Thus $\bar\omega(f, g)$ can be thought of as the primary obstruction to separating f and g, analogous to the primary obstruction to identifying f and g, namely the primary difference $\bar\alpha(f, g)\in H^q(X; H_q(Y))$, where we assume that $\pi_q(Y)=H_q(Y)$, $q>1$, is the first non-vanishing homotopy group of Y, and is stable under $\pi_1(X)$ [Steenrod, The topology of fibre bundles, Princeton, 1951; MR 12, 522].

Now suppose $K=K^N$, $N=q+n$, and consider the problem of extending over K a cross-section f given over K^{N-1}. The homomorphism $\pi_{N-1}(Y)\to H_q(Y)$ for the manifold Y [Gysin, Comment. Math. Helv., 14 (1941), 61–122; MR 3, 317] gives rise to an obstruction

$$\Gamma(f)\in H^{N-1}(X; H_q(Y)),$$

which must vanish if f is to be extended. If $\Gamma(f)\neq 0$, and one is seeking for another cross-section g over K^{N-1} which can be extended, one naturally asks if $\Gamma(f)=\Gamma(g)$. To this end the author proves the main formula

$$\Gamma(f)-\Gamma(g)=\bar\alpha(f, g)\cup\bar\omega(f, g),$$

in which the right-hand side clearly vanishes if f, g either coincide over K^q or separate over K^n, or if $H^q(X)$ or $H^n(X)$ vanishes, or if X is a manifold with zero nth Betti number.

The formula first appeared in 1950 [Hopf, Colloque de topologie (espaces fibrés), Bruxelles, 1950, Thone, Liège, 1951, pp. 9–14; MR 12, 847], but this is the first published proof. It has been used by Hirzebruch, Kundert and the author. *E. C. Zeeman* (Cambridge, England).

Blanchfield, Richard C. Intersection theory of manifolds with operators with applications to knot theory. Ann. of Math. (2) 65 (1957), 340–356.

Let $\mathfrak{M}$ be an oriented manifold with boundary and let G be a free abelian multiplicative group of covering transformations of a covering space $\mathfrak{M}^\dagger$ of $\mathfrak{M}$. The automorphism $\gamma\to\gamma^{-1}$ of G extends to a unique automorphism $\alpha\to\bar\alpha$, called conjugation, of the integral group ring R of G. The homology groups of $\mathfrak{M}^\dagger$ are R-modules. Reidemeister has defined [Monatsh. Math. Phys. 48 (1939), 226–239; MR 1, 105] an intersection S, which pairs the homology modules of dual dimension to R, and a linking V, which pairs the torsion submodules of dual dimension to R_0/R, where R_0 is the quotient field of R. The following generalizations of the Poincaré-Lefschetz duality theorems are proved: (1) Let π be zero or a prime of R and let m be any positive integer. Then the Betti modules of dual dimensions with coefficients modulo π^m and $\bar\pi^m$ respectively are primitively paired by S to R/π^m. (2) The dual-dimensional torsion modules of weakly bounding cycles modulo those that bound weakly modulo π^m for all primes π and all positive integers m are primitively paired by V to R_0/R.

The elementary divisor ideals of dual torsion modules are shown to be conjugates of one another. This result is applied to the maximal abianel covering of a link in a closed 3-manifold: The elementary divisor ideals Δ_i of the 1-dimensional torsion module are symmetric in the sense that $\bar\Delta_i=\Delta_i$. This extends results of Seifert [Math. Ann. 110 (1934), 571–592] and Torres [Ann. of Math. (2) 57 (1953), 57–89; MR 14, 574] about the symmetry of the Alexander polynomial (in the author's notation, a generator of Δ_0) of a knot or link in Euclidean 3-space. *R. H. Fox* (Princeton, N.J.).

Gamkrelidze, R. V. Chern's cycles of complex algebraic manifolds. Izv. Akad. Nauk SSSR. Ser. Mat. 20 (1956), 685–706. (Russian)

This paper brings the detailed proofs for results announced earlier [Dokl. Akad. Nauk SSSR (N.S.) 90 (1953), 719–722; MR 15, 459]. A large part of the paper is taken up by the construction of the cycles of the complex Grassmann manifolds, and related flag manifolds, utilizing Ehresmann's cell division. *H. Samelson.*

Fet, A. I. Involutory mappings and coverings of spheres. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. 1 (1956), 55–71. (Russian)

This paper gives detailed proofs of the results of an earlier paper [Dokl. Akad. Nauk SSSR (N.S.) 95 (1954), 1149–1151; MR 16, 61]. The main theorem (incorrectly stated on p. 55) asserts that if θ is a map of S^n to itself of period 2 and if $\{F_1, F_2, \cdots, F_{n+1}\}$ is a covering of S^n by closed sets, then at least one of the sets F_i contains a pair of points a, θa.

Clearly we may suppose θ to be fix-point free; let Π^n be the topological manifold obtained from S^n by identifying a, θa for all $a\in S^n$. A point of more general interest in the proof is the demonstration that the Čech-Alexandrov and singular homology theories coincide on Π^n. *P. J. Hilton* (Manchester).

See also: Hilton, p. 10; Eckmann, p. 14; Eichler, p. 18; Bernstein, p. 49; Wagner, p. 49; Calabi, p. 62; Frölicher and Nijenhuis, p. 62.

GEOMETRY

Geometries, Euclidean and other

Bini, Umberto. Esiste ancora una geometria razionale? Archimede 9 (1957), 6–7.

An expository note on the significance of Gödels' theorem for a geometric system.

★ Brusotti, Luigi. A proposito di una caratterizzazione della retta negli spazi euclidei. Scritti matematici in onore di Filippo Sibirani, pp. 33–39. Cesare Zuffi, Bologna, 1957.

A brief, historical account with special emphasis on the characterization by H. Mohrmann of the straight line in euclidean space as the curve for which "every part is similar to every other part".

Straszewicz, S. Sur la trigonométrie de Lobatchevsky. Ann. Polon. Math. 3 (1957), 225–239.

The author gives new proofs of the fundamental theorems of hyperbolic trigonometry, which are based only on the theorem that the sum of the angles of a triangle is $<\pi$, the axiom of Archimedes and certain theorems which can be deduced from axioms I_{1-3}, II and III of Hilbert's system. *L. A. Santaló.*

★ **Норден, А. П. [Norden, A. P.] Об основаниях геометрии. [Foundations of geometry.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 527 pp. 18.70 rubles.

This is a collection of classical articles on the geometry of Lobačevskiĭ and the development of its ideas. There is a general introduction by A. P. Norden and an appendix with bibliographical information by I. N. Bronšteĭn. The essays are grouped in three sections. In the first come three articles by Lobačevskiĭ, one by Bolyai and one by Gauss. The second section, entitled "Foundations of the theory of surfaces and interpretation of the geometry of Lobačevskiĭ", contains articles by Gauss, Minding, Beltrami, Hilbert, Cayley, Klein and Poincaré. In the third section, entitled "Development of the ideas of the geometry of Lobačevskiĭ", are articles by Riemann, Beltrami, Helmholtz, Lie, Poincaré, Klein, Hilbert, Kagan and Cartan.

Jurkat, Wolfgang B. Zur Grundlegung der Geometrie. Jber. Deutsch. Math. Verein. **59** (1957), Abt. 1, 87–92.

Where G. Pickert in "Analytische Geometrie" [Akademische Verlagsgesellschaft, Leipzig, 1953; MR **15**, 339; see also Math.-Phys. Semesterber. **4** (1955), 239–249] presents a foundation of geometry which presupposes an algebraic structure, an attempt is made in this paper to lay a foundation of geometry and analysis in an analogous way so that the mutual correspondence of both structures becomes evident. Fundamental concepts are points $\mathfrak{P}$ and vectors (translations) $\mathfrak{B}$; vectors are mappings of the $\mathfrak{P}$ into themselves. Four axioms on $\mathfrak{B}$ establish the vectors as an Abelian group of such mappings. Then scalars (dilations) $\mathfrak{S}$ are introduced by a set of axioms as mappings of $\mathfrak{B}$ into themselves; they form a ring to modulus $\mathfrak{B}$. Next follows a set of axioms for the concept of directions. The scalars now form an ordered domain $\mathfrak{S}$. The fourth concept introduced by axioms is that of the unit vector. Now a vector space $\mathfrak{B}$ over $\mathfrak{S}$ with inner product is obtained; conversely, if such a space is given, then the given axioms can be realized. Further construction of the geometry can now be interpreted simultaneously in a geometrical or analytical way. Special cases are Euclidean space of n dimensions and real Hilbert space.

S. R. Struik (Cambridge, Mass.).

Lenz, Hanfried. Axiomatische Bemerkung zur Polarentheorie. Math. Ann. **133** (1957), 39–40.

Verfasser hatte an früherer Stelle [Math. Ann. **128** (1954), 363–372; MR **16**, 739] gezeigt, daß in einem projektiven Raum der endlichen Dimension $n>2$ eine verallgemeinerte Hermitesche Polarität π durch folgende 3 Forderungen gekennzeichnet ist: 1) eindeutige Bestimmtheit der Polarhyperebene $\pi(A)$ zum Punkt A, 2) jede Hyperebene hat höchstens einen Pol, 3) aus $A\subset\pi(B)$ folgt $B\subset\pi(A)$. In der vorliegenden Note wird bewiesen, daß das Axiom 2) bereits aus den übrigen beiden folgt.

W. Burau (Hamburg).

★ **Freudenthal, Hans. La topologie dans les fondements de la géometrie.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 178–184. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Exposé (sans indications bibliographiques) de l'état atteint, fin 1953, grâce aux travaux de l'A. et d'autres auteurs cités, du problème suivant: Quelles propriétés (de caractère topologique, notamment) un espace dans lequel on peut construire une géométrie projective doit-il nécessairement avoir? Après un rappel des travaux de von Staudt à Hilbert (corps algébrique attaché à une géométrie arguésienne, première construction d'un plan non-arguésien), l'A. évoque d'abord un premier aspect, considéré par le rapporteur, puis par l'A.: conséquences résultant de l'existence dans l'espace (et plus spécialement, dans le plan projectif) d'une famille de sous-ensembles privilégiés (droites projectives) vérifiant certaines relations (les axiomes d'incidence de la géométrie projective). Moyennant des hypothèses de régularité, le rapporteur a démontré que si le plan et les droites projectives sont des variétés topologiques, alors les droites sont homéomorphes à des sphères dont la dimension est une puissance de 2; l'A. établit que si le plan projectif est un espace compact, et les droites des sous-ensembles fermés connexes, alors tout sous-ensemble propre fermé de la droite est contractile en un point (sur la droite). Outre les espaces projectif réels, complexes et quaternioniens, il existe un plan projectif des octaves de Cayley (à 16 dimensions topologiques) qu'on peut construire, soit en adjoignant une droite à l'infini au plan affin des octaves construit par Mlle R. Moufang, soit à partir de la fibration en S^7 de la sphère S^{15} à 15 dimensions (par une méthode due au rapporteur), soit encore au moyen de l'algèbre exceptionnelle de P. Jordan (matrices hermitiennes avec des octaves pour coefficients); cette dernière interprétation permet à l'A. d'interpréter les groupes exceptionnels E_6 et F_4 au moyen de la géométrie du plan des octaves; l'A. signale que quelques-uns des résultats qu'il a publiés à ce sujet avaient déjà été annoncés antérieurement par P. Jordan.

Une autre méthode inspirée par le groupe des projectivités repose sur la mobilité des figures, d'après Riemann, Helmholtz et Lie, jusqu'à Kolmogorov. Une classification des groupes triplement transitifs (qui, dans les espaces localement compacts, métrisables et non totalement discontinus, coïncident avec le groupe projectif de la droite réelle ou complexe) a été faite par Kérékjarto, Tits et l'A. Tits et ensuite A. Borel ont étudié les groupes doublement transitifs. Un dernier aspect fait intervenir les isométries, envisagé par Busemann, Wang et Tits; ce dernier a donné en même temps la démonstration des résultats de Kolmogorov. La méthode repose sur la recherche des groupes de Lie ayant certains groupes irréductibles comme groupe de stabilité.

G. Hirsch.

McCarthy, J. P. The cissoid of Diocles. Math. Gaz. **41** (1957), 102–105.

Dörrie, H. Der Satz von Ptolemäus nebst Anwendungen. Archimedes **8** (1956), 33–40.

Die erweiterte Fassung wird bewiesen: Ein Viereck ist dann und nur dann ein Kreisviereck, wenn eins seiner drei Gegenseitenprodukte der Summe der anderen beiden gleicht; im Nichtkreisviereck ist jedes Gegenseitenprodukt kleiner als die Summe der beiden andern.

Aus der Einleitung.

Tenca, Luigi. Proprietà della semisfera. Period. Mat. (4) **34** (1956), 278–283.

Various properties of the upper half of the figure of Archimedes, i.e. of a hemisphere inscribed in a cylinder.

Thébault, Victor. Géométrie et mécanique. Mathesis **66** (1957), 28–34.

Generalizations of the lemmas of Archimedes about the shoemaker's knife, i.e. areas bounded by various arcs of semicircles.

Kagan, V. F. On a geometrical problem in the theory of cutting tools. Trudy Sem. Vektor. Tenzor. Anal. **10** (1956), 23–29. (Russian)

Skorobogat'ko, V. Ya. A bisectorial surface and its properties. Dopovidi Akad. Nauk Ukrain. RSR. **1956**, 419–422. (Ukrainian. Russian summary)

Goormaghtigh, R. Sur les courbes parallèles aux épi- et hypocycloïdes. Mathesis **65** (1956), 429–430.

The equation $x=nt+t^n$, where t is a variable complex number of modulus 1, represents an epicycloid or hypocycloid according as n is positive or negative. The author shows that the parametric equation

$$x=nt+ct^{\frac{1}{2}(n+1)}+t^n,$$

where c is a constant, real or complex, represents a curve parallel to an epicycloid or hypocycloid.

The latter equation, for $n=-3$, is used to prove that the envelope of the Simson line of a variable point of a circle for an inscribed quadrilateral is an oblique astroid. The author proved this proposition before in a different way [Amer. Math. Monthly **49** (1942), 174–181, p. 177; MR **3**, 251] (the reference given by the author is inexact). The definition of the Simson line of a cyclic quadrilateral used in this proposition is the one given by E. M. Langley [Educational Times **41** (1889), 490].

N. A. Court (Norman, Okla.).

Behrens, D. J. Problem. Math. Gaz. **41** (1957), 101.

A partial solution of an interesting geometrical problem about chords in a circle which can arise in duplicate bridge competitions.

Court, Nathan Altshiller. On three intersecting circles. Math. Student **24** (1956), 217–226 (1957).

Three mutually intersecting circles (A), (B), (C) have one circle (M) orthogonal to them and their three pairs of intersecting points determine eight "triangles of intersection" if the condition is imposed to select a vertex from each pair. Paired off judiciously each of the four pairs of triangles can be shown to be perspective (Desargues theorem). Their four axes of perspectivity form a complete quadrilateral, called here "quadrilateral of intersection," so that the pairs of opposite vertices of "the quadrilateral of intersection" for (A), (B), (C) are pairs of conjugate points with respect to (M). This is proven with reference to L. Cremona, "Elements of projective geometry", 3rd ed. [Oxford, 1913, § 336]. Properties having a bearing on the real or imaginary character of (A), (B), (C) are discussed, or referred to N. A. Court, Amer. Math. Monthly **62** (1955), no. 7, part II, 59–56 [MR **17**, 398].

S. R. Struik (Cambridge, Mass.).

Tallini, Giuseppe. Sulle k-calotte di uno spazio lineare finito. Ann. Mat. Pura Appl. (4) **42** (1956), 119–164.

Let S_n be a projective space of dimension n over a GF(q), where $q=p^n>2$ (p prime); denote by $\mathcal{C}(k)$ any proper subset of S_n, consisting of k points, and such that either (i) no three of its points are collinear or (ii) for any three collinear points of $\mathcal{C}(k)$ the whole line joining them belongs to $\mathcal{C}(k)$. The present paper establishes a number of properties of the sets $\mathcal{C}(k)$ (called "k-calotte") [the fundamental case when $n=2$, $p>2$, $k=q+1$ has previously been studied by B. Segre, Canad. J. Math. **7** (1955), 414–416; other cases are dealt with in B. Segre, Ann. Mat. Pura Appl. (4) **39** (1955), 357–379; MR **17**, 72, 776]. The main results obtained are as follows.

If $n\geqq 3$ and $\mathcal{C}(k)$ presents the alternative (i), then $k\leqq q^{n-1}$ [for $n=3$, cf. B Qvist, Ann. Acad. Sci. Fenn. Ser. A. I. no. **134** (1952); MR **14**, 1008]. Next we do not exclude alternative (ii), and suppose $n\geqq 3$, $k\geqq 1+q+q^2+\cdots+q^{n-1}$; then $\mathcal{C}(k)$ is necessarily of one of the following types. I: $\mathcal{C}(k)=S_{n-1}+S_t$, where S_{n-1}, S_t are subspaces of S_n and $-1\leqq t\leqq n-1$. II: If $p>2$ and we are not in case I, $\mathcal{C}(k)$ is a nonspecialized quadric of a space of even dimension, or a cone obtainable from such a quadric by projection, or a nonspecialized quadric of hyperbolic type of a space of odd dimension, or a quadric cone of hyperbolic type. III: If $p=2$ and we are not in case I, $\mathcal{C}(k)$ either can be obtained from a quadric of one of the types II by adding to it the points of a subspace S_t of S_n, or is a cone projecting from an S_{n-3} of S_n the points of a $\mathcal{C}(q+1)$ or $\mathcal{C}(q+2)$ of an S_2 of S_n skew to S_{n-2}. *B. Segre* (Rome).

Ram, Sahib. Oblique co-ordinates. Math. Student **24** (1956), 235–239 (1957).

The object of this paper is to discuss the curvature of curves and find curves for which the tangent, subtangent, normal or subnormal is constant when the axes of reference are inclined at an angle ω. *Author's summary.*

Steinberg, R. Generalizations of the theorem of Chasles. Amer. Math. Monthly **64** (1957), 352–353.

The author considers an n-dimensional vector space V together with a bilinear symmetric scalar product (x, y) defined for pairs of vectors x, y. In terms of a covariant basis $v_1, \cdots, v_n$ and a contravariant basis $v^1, \cdots, v^n$, connected by the relations $(v_i, v^j)=\delta_i^j$, any vector x has covariant components $x_i=(x, v_i)$ and contravariant components $x^j=(x, v^j)$, such that $x=\sum x_iv^i=\sum x^jv_j$ and consequently $(x, y)=\sum x_iy^i=\sum x^iy_i$. The author has noticed a geometrical interpretation for the identity

$$\sum (x_iy^i-x^iy_i)=0.$$

Since $x_iy^i-x^iy_i=0$ is the condition for the plane Π spanned by x and y to contain a non-zero vector orthogonal to the plane Π_i spanned by v_i and v^i, we have the following theorem. If Π contains $n-1$ vectors orthogonal to the respective planes $\Pi_2, \cdots, \Pi_n$, it also contains a vector orthogonal to Π_1.

He gives a similar result when the v's are replaced by two sets of more than n vectors, and another when n becomes infinite so that V is real Hilbert space.

H. S. M. Coxeter (Toronto, Ont.).

Sommer, F. Analytische Geometrie im C^n. Schr. Math. Inst. Univ. Münster no. **11** (1957), ii+48 pp.

An expository lecture whose purpose is to acquaint the reader with "die Beziehungen, die sich zwischen dem C^n und dem ihm zugeordneten reellen Raum R^{2n} ergeben. Hierbei war es ein besonderes Anliegen, wesentliche Begriffe wie 'komplexe Strukturen' und 'hermitesche Formen und Metriken' darzustellen."

Lasley, J. W., Jr. On degenerate conics. Amer. Math. Monthly **64** (1957), 362–364.

Three criteria for determination of the type of a degenerate conic.

Court, N. A. Three hyperbolas associated with a triangle. Amer. Math. Monthly **64** (1957), 241–247.

A conic (q) is circumscribed about a triangle (T). The tangents to (q), at the vertices of (T), determine the triangle (T'). (T) and (T') are in perspective; pole M and

polar m of the perspectivity (called Lemoine point and Lemoine axis of (T) for the conic (q)) are also trilinear pole and polar for (T) and (T') both. If a point describes a straight line u, its isotomic conjugate point for a triangle (T) describes a conic (q), circumscribed about (T). The Lemoine axis of (T) for (q) is the isotomic transversal u' of u for (T). The requirement that u and its isotomic conjugate transversal coincide produces four such exceptional cases: u is either the line i at infinity or one of the three sides of the medial triangle of (T). Their respective isotomic conics are the circumscribed minimum (Steiner) ellipse and three hyperbolas $(H_{a,b,c})$ circumscribed about (T). The three centers of the $(H_{a,b,c})$ form a triangle (ω), homothetic to (T). Moreover, the line connecting the centers of any two of the three hyperbolas is tangent to the third hyperbola. A median of (T) passes through the center G of the Steiner ellipse, and two of the medians are tangent to (H_a). Triangle (ω) may be treated anew like (T) and its characteristic three hyperbolas derived, and so ad infinitum. These and other theorems show that there is here a fertile field of research. *S. R. Struik* (Cambridge, Mass.).

Miller, Robert C., Jr. Foci of the conics on a cone. Math. Mag. **30** (1957), 193–204.

Dandelin proved [Ann. Mat. Pures Appl. **15** (1824–1825), 387–396]: Two spheres inscribed in a right circular cone and tangent to a plane have two points of tangency which are foci of the conic determined by the intersection of the plane and cone. The author proves: When a pencil of such planes intersects the cone, the locus C_4 of the foci is in general a quartic curve, the basis curve of a pencil of quadric surfaces; it is rational with a nodal double point (the vertex of the cone). It is a plane curve when the axis of the pencil is perpendicular to the axis of the cone. It degenerates into a space cubic and a straight line when the axis of the pencil is tangent to the cone. The different curves obtained in the case of a plane curve are drawn and discussed. *S. R. Struik.*

Godeaux, Lucien. Remarque sur les couples de congruences W ayant une nappe focale commune. Bull. Soc. Roy. Sci. Liège **25** (1956), 514–519.

In previous work, the author studied the configuration formed by two congruences of lines W which possess a common focal surface S. In particular, various properties of a family of quadrics associated with these two congruences W have been obtained. In the derivations of these properties, use was made of the representation of the focal nappes of these two congruences W upon the hyperquadric Q of Klein in a space S_5 of five dimensions. In the present work, the author gives a new development of the same results. The method is based on the idea of immersing all the configurations in a linear space S_7 of seven dimensions. By these methods, some properties of a sequence of Laplace are obtained. *J. De Cicco.*

Sobczyk, Andrew. Simple families of lines. Pacific J. Math. **6** (1956), 541–552.

Etude de certaines familles de droites dans E_{n+1}, en particulier quant à la propriété de contenir une seule droite dans chaque direction, ou de recouvrir l'espace, ou l'extérieur d'une sphère.

Considèrant deux $E_n(\subset E_{n+1})$ parallèles, x un point de l'un de ces E_n, y un point de l'autre, on étudie et on classe en particulier les familles définies par: $y=Tx+u$, où T est une transformation affine, et u un vecteur donné de E_n; les différents éléments simples de la forme canonique d'une transformation T donnent lieu à l'introduction de familles de droites particulières qui sont ensuite composées pour retrouver la famille donnée. On esquisse enfin une généralisation aux familles de droites contenues dans un espace de Banach. *J. Favard* (Paris).

Kuiper, N. H. A real analytic non-desarguesian plane. Nieuw Arch. Wisk. (3) **5** (1957), 19–24.

Verf. gibt ein reell analytisches Modell einer nicht-Desarguesschen, sogar nicht-harmonischen projektiven Ebene an, in der die Pseudopunkte die Punkte einer affinen Ebene sind, die Pseudogeraden die Geraden $x=$const, $y=$const und die algebraischen Kurven 3. Grades
$$x=p+y\operatorname{cotg}\varphi+\frac{\rho y^2\sin 2\varphi}{1+y^2}\quad(\rho=0,1).$$
In homogenen Koordinaten $X:Y:Z=x:y:1$ lassen sich die Pseudogeraden darstellen als die Gesamtheit der Kurven
$$(\lambda X+\mu Y+\gamma Z)(Y^2+Z^2)(\lambda^2+\mu^2)+\rho\lambda^2\mu Y^2Z=0$$
nach Herausnahme des isolierten Doppelpunktes $Y=Z=0$ im Falle $\lambda\mu\neq0$. Die numerische Wahl des Parameters ρ sichert insbesondere die Gültigkeit der projektiven Verknüpfungssätze. *R. Moufang* (Frankfurt am Main).

Ladopoulos, Panaiotis D. Généralisation du théorème de Desargues-Sturm. Bull. Soc. Roy. Sci. Liège **26** (1957), 16–18.

Primrose, E. J. F. The representation of projectivities. Math. Gaz. **41** (1957), 117–119.

A one-to-one correspondence is set up between the family of projectivities of a 1-dimensional projective space and the points of a 3-dimensional space. Some standard theorems about such projectivities are then interpreted in the 3-dimensional space.

Manevič, V. A. On a representation of the elements of systems of collineations in the plane and in space as products of two polarities and on certain properties of collineation connected with this question. Mat. Sb. N.S. **41(83)** (1957), 221–230. (Russian)

Three non-collinear points U_1, U_2, U_3 may be considered as the double points of a net of collineations in the plane. They may also be taken for the vertices of the common polar triangle of the pencil of conics (K^2) passing through the vertices M, H, K, P of a complete quadrangle having $U_1U_2U_3$ for diagonal triangle, one of the four vertices, say M, having been chosen arbitrarily.

If a, a' are an arbitrary pair of lines in the plane, their poles for (K^2) lie, respectively, on two conics passing through U_1, U_2, U_3 and meeting in a fourth point F. The pencil (K^2) includes two conics (K_1^2), (K_2^2) such that the polars of F with respect to them coincide with the lines a, a', respectively. From the chain
$$a-(K_1^2)-F-(K_2^2)-a'$$
the author concludes that the collineation $(U_1U_2U_3, aa')$ is the product of the two polarities (K_1^2), (K_2^2) [cf. Veblen and Young, Projective geometry, vol. I, Ginn, Boston, 1910, p. 265, Th. 4; H. S. M. Coxeter, The real projective plane. McGraw-Hill, New York, 1949, p. 67; MR **10**, 729]. Any net of collineations may thus be considered as the polar system of a certain pencil of conics (K^2), and that pencil is not unique (due to the arbitrary choice of the point M).

Analogous considerations are presented for collineations in space in connection with pencils of quadric surfaces.

N. A. Court (Norman, Okla.).

Kolobov, P. G. Kirkman lines and cubic curves. Rostov. Gos. Ped. Inst. Uč. Zap. no. 3 (1955), 45–51. (Russian)

From an article contributed by K. A. Andreev [Soobšč. Har'kov. Mat. Obšč. (2) 1 (1889), 277–280] the author quotes the following two propositions: (a) If a heptagonis circumscribed about a conic and each vertex is joined to the third following vertex (in the same sense of rotation), the points of intersection of each line with its consecutive line lie on a conic. (b) With seven tangents to a conic it is possible to form 360 heptagons to each of which correspond seven points on a conic (by Prop. (a)).

In his study of the "Andreev configuration" generated by Prop. (b) the author makes use of Pascal's Hexagrammum mysticum, and particularly of Kirkman's theorem: "The sixty Brianchon points (denoted by S) of a hexagon circumscribed about a conic lie by threes on sixty (Kirkman) lines (denoted by h), and through each S point pass three (Kirkman h) lines.

It is "perfectly evident" that each point generated in Prop. (a) is a Brianchon point S of some hexagon obtained by omitting one of the seven given tangents, hence the Andreev configuration consists of 7 groups, each containing sixty S points, so that the configuration consists of $60\times 7=420$ S points lying by sevens on 360 conics.

The author quotes from one of his papers [same Zap. 1953, no. 2, 55–62 (not accessible to this reviewer); MR 17, 884] the following proposition: "The 420 S points lie by 10 on 252 conics (denoted by β), each point belonging to six of the conics."

In the paper under review he adds that: "Each β conic contains, in addition to the 10 points just mentioned, 10 other points of intersection of the Kirkman h lines", and he closes his article with the theorem: "There are 504 curves of the third order which contain each 15 points of intersection of pairs of Kirkman h lines." *N. A. Court.*

★**Pavlíček, Jan B. Základy neeukleidovské geometrie Lobačevského. [Elements of the non-Euclidean geometry of Lobachevski.]** Přírodovědecké Vydavatelství, Praha, 1953. 223 pp. 136 Kčs.

Fladt, Kuno. Die allgemeine Kegelschnittsgleichung in der ebenen hyperbolischen Geometrie. J. Reine Angew. Math. 197 (1957), 121–139.

Eine wirklich geometrische und elementare Methode, die Kegelschnitten in der hyperbolischen Ebene zu klassifizieren, sei hier dargelegt. Auf elementargeometrischem Wege bringt sie die allgemeine Kegelschnittsgleichung auf eine Normalform, welche die Art der Kegelschnitte mit wenig elementarer Rechnung zu erkennen gestattet.

Aus der Einleitung.

Fladt, Kuno. Die allgemeine Gleichung der Flächen zweiten Grades in der hyperbolischen Geometrie. J. Reine Angew. Math. 197 (1957), 140–161.

Im folgenden soll die vollständige Klassifikation auch aller Nebenfälle (der Flächen zweiten Grades) dadurch auf elementarem Wege geleistet werden, dass wir mit Hilfe einfacher hyperbolischer Bewegungen geeignete Normalformen der Flachengleichung herstellen.

Aus der Einleitung.

★**Dieste, Rafael. Nuevo tratado del paralelismo. [New treatise on parallelism.]** Colección Oro de Cultura General, 157. Editorial Atlantida, S. A., Buenos Aires, 1956. 186+xii pp.

An expository account, with historical and philosophical emphasis.

Mendonça de Albuquerque, Luís. Observation on some problems of perspective. Rev. Fac. Ci. Univ. Coimbra 24 (1955), 8–11. (Portuguese)

Pour simplifier la résolution graphique de certains problèmes de Perspective proprement dite et de Perspective Cavalière, l'auteur de cet article propose l'adjonction d'un système de Monge, c'est-à-dire un système formé de deux plans orthogonaux, qui se fait couramment dans la Perspective Appliquée. *F. Şemin* (Istanbul).

See also: Grosheide, p. 1; Tits, p. 44; Gale, p. 57; Rusieshvili, p. 58; Glagolev, p. 77.

Convex Domains, Integral Geometry

Hadwiger, H.; et Debrunner, H. Choix de quelques problèmes de géométrie combinatoire dans le plan. Enseignement Math. (2) 3 (1957), 35–70.

A translation from the original German of the article reviewed in MR 17, 887.

★**Люстерник, Л. А. [Lyusternik, L. A.] Выпуклые фигуры и многогранники. [Convex figures and polyhedra.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 212 pp. 2.95 rubles.

The six chapter-headings are: convex figures and bodies and their support lines and planes; centrally symmetric convex figures; convex polyhedra; linear systems of convex bodies; theorems of Minkowski and Alexandrov (this chapter was written by Alexandrov); supplementary material. Visualizability is emphasized throughout. The entire book can be read with very little mathematical background, the first three chapters with almost none at all.

★**Gale, David. Neighboring vertices on a convex polyhedron.** Linear inequalities and related systems, pp. 255–263. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00

Remarking that Kuhn has discovered (oral communication) a polytope in eleven dimensions such that every two vertices form an edge (so that there are no "diagonals"), the author proves that such a phenomenon already arises in four dimensions (for any given number of vertices). In fact, he gives an ingenious but indirect proof that for any two integers $m>0$, $n>2m$, there exists a $2m$-dimensional polytope having n vertices of which every m form a simplex which is an $(m-1)$-dimensional element (e.g., an edge when $m=2$).

H. S. M. Coxeter (Toronto, Ont.).

Hadwiger, H. Ausgewählte Probleme der kombinatorischen Geometrie des Euklidischen und sphärischen Raumes. Enseignement Math. (2) 3 (1957), 73–75.

This is a note concerning combinatorial-geometric theorems of the Helly type in k-space and the Horn type on the k-sphere. The author notes certain connections as well as differences between such spherical and space type

theorems and concludes with the following general problem: Given a family of convex bodies in k-space (the k-sphere) such that for any p of the bodies some collection of q of these has a nonempty intersection, $p \geqq k \geqq q+1$, does there exist a number $N=N(p, q; k)$ $(M=M(p, q; k))$ such that there exists a set of N (M) points which meets every set of the family? Helly's Theorem asserts that $N(k+1, k+1, k)=1$; Horn's implies $M(k+1, k+1, k)=2$. For $k=1$ the answer is $N(p, q; 1)=p-q+1$ and $M(p, q; 1) \leqq p-q+2$. *D. Gale* (Santa Monica, Calif.).

Redheffer, Raymond. A curious formula for distance. Amer. Math. Monthly **64** (1957), 195–196.

The following theorem is proved.

Let R be a convex region in (x, y, z) space containing the points P and Q. Then the distance from P to Q is

$$d(P, Q)=\sup_f \inf_{(x,y,z)} \{f(P)-f(Q)\}(f_x^2+f_y^2+f_z^2)^{-\frac{1}{2}}$$

where the sup is taken over all non-constant functions differentiable in R.

Wendel, J. G. Comment on "The distance to the origin of a certain point set in E^n". Proc. Amer. Math. Soc. **8** (1957), 413–414.

This is a shorter proof using the theory of orthogonal polynomials of the result of Karush and Wolfsohn [same Proc. **6** (1955), 323–332; MR **16**, 1047] concerning the minimum of $\sum_{i=0}^{m} a_i^2$ subject to the conditions $\sum_{i=0}^{m} a_i=1$, $\sum_{i=0}^{m} i^k a_i=0$ $(k=1, 2, \cdots, r)$, where r is a given integer with $0 \leqq r < m$. *F. F. Bonsall* (Newcastle-on-Tyne).

Besicovitch, A. S. A net to hold a sphere. Math. Gaz. **41** (1957), 106–107.

The length of string needed to construct a net around a unit sphere, so that the sphere cannot slip out of it, is greater than 3π but can be made as near 3π as we like.

Rusieshvili, C. I. Some extremal problems of the intrinsic geometry of polyhedrons in Lobatchevsky's space. Vestnik Leningrad. Univ. **12** (1957), no. 1, 76–79, 209. (Russian. English summary)

Warncke, Donald; and Supnick, Fred. On the covering of E_n by spheres. Proc. Amer. Math. Soc. **8** (1957), 299–303.

The paper is concerned with the covering of E_n provided by the set S_n of closed solid n-spheres with centers at the points $(x_1, x_2, \cdots, x_n)$, where x_i is an integer $(i=1, 2, \cdots, n)$, and radius $(n^{\frac{1}{2}})/2$. It is shown that each point of E_n is a point of some n-sphere of S_n with center at $(i_1, i_2, \cdots, i_n)$ with $i_1+i_2+\cdots+i_n=0 \bmod([n/4]+1)$, where $[x]$ denotes the greatest integer less than or equal to x. Thus for $n>3$, E_n is covered by a proper subset of S_n, which is not the case for $n=1, 2, 3$.

The argument is entirely elementary. The authors raise the question of determining for what values of n (if any) the number $[n/4]+1$ may be replaced by a larger one.
L. M. Blumenthal (Columbia, Mo.).

Differential Geometry

★ **Norden, A. P. Differentialgeometrie. Teil I.** VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. viii+135 pp.

A translation by Rolf Sulanke of the original Russian book published by Gosudarstv. Uč.-Ped. Izdat., Moscow, 1948.

Vidal Abascal, Enrique. Present-day methods and problems of differential geometry. Rev. Mat. Hisp.-Amer. (4) **17** (1957), 38–58. (Spanish)

An expository article.

Cochină, A. On a formula of the theory of surfaces. Gaz. Mat. Fiz. Ser. A. **11** (1956), 570–574. (Romanian)

Proof of the well-known Weingarten formulae in the elementary theory of surfaces which the author claims to be new. *R. Blum* (Saskatoon, Sask.).

Gheorghiu, Gh. Th. Sur une interprétation métrique de quelques invariants affins. Com. Acad. R. P. Romîne **5**, 325–331 (1955). (Romanian. Russian and French summaries)

On the basis of a remark by G. Tzitzéica [Ann. Sci. Ecole Norm. Sup. (3) **28** (1911), 9–32], the author gives a geometrical interpretation of certain invariants (under affine unimodular transformations) of 1) plane and 2) space curves. These involve in case 1) the metric radius of curvature and in case 2) the metric radius of torsion.
R. Blum (Saskatoon, Sask.).

Vančura, Zdeněk. Les surfaces focales des congruences de sphères. Časopis Pěst. Mat. **80** (1955), 317–327. (Czech. Russian and French summaries)

Let $\mathbf{c}$ be a linear complex of L-spheres (Lie spheres), $\mathbf{p}$ an arbitrary sphere of $\mathbf{c}$, $\mathbf{q}$ a generic L-sphere of $\mathbf{c}$, $P(\mathbf{q})$, $(p(\mathbf{q}))$ the common point (tangent plane) of $\mathbf{p}$ and $\mathbf{q}$. The correlation $P \rightarrow p$ is termed a contact correlation induced on $\mathbf{p}$ by $\mathbf{c}$. If $\mathbf{p}=\mathbf{p}(t)$ is a canal surface S, then every complex of the pencil $\mathbf{c}=\lambda \mathbf{p}(t_0)+\mu(d\mathbf{p}/dt)_{t_0}$ induces on $\mathring{\mathbf{p}}=\mathbf{p}(t_0)$ the same contact correlation, termed the characteristic correlation (Ch-correlation). If $\mathbf{p}=\mathbf{p}(\eta^{\mathrm{I}}, \eta^{\mathrm{II}})$ is a congruence C_2 of L-spheres, $\eta^\alpha=\eta^\alpha(t)$ a canal surface in C_2, then every complex of the pencil

$$\mathbf{v}=\lambda\mathring{\mathbf{p}}+\mu[(d\eta^\alpha/dt)(\partial\mathbf{p}/\partial\eta^\alpha)]_{t_0}$$

induces on the sphere $\mathring{\mathbf{p}}=\mathbf{p}(\eta^{\mathrm{I}}(t_0), \eta^{\mathrm{II}}(t_0))$ the same Ch-correlation. The locus of points $P(\mathbf{v})$ is a circle in a plane $\pi(\mathbf{v})$ (on $\mathring{\mathbf{p}}$). For every canal surface through $\mathring{\mathbf{p}}$ in C_2 we obtain in this way a plane $\pi(\mathbf{v})$. The set of all these planes is a pencil Π, projective with the pencil of the vectors $v^\alpha=(d\eta^\alpha/dt)_{t_0}$. The axis of Π intersects $\mathring{\mathbf{p}}$ in two points F_h $(h=1, 2)$. The tangent plane of F_h to $\mathring{\mathbf{p}}$ is denoted by f_h. [For these notions see Hlavatý, Věstnik Kr. České Společnosti Nauk. Třída Mat.-Přírodoved. **1941**, no. 6; Rozpravy II. Třídy České Akad. **51** (1941), no. 33; MR **7**, 483; **13**, 1138; **9**, 64.]

The author applies previous results to the investigation of the focal points F_h and focal planes f_h. He proves first that the surface element (F_h, f_h) on $\mathring{\mathbf{p}}$ is common to all canal surfaces of C_2 through $\mathbf{p}$. Then he derives the equation of $\pi(\mathbf{v})$. This enables him to find parametric equations of the hth focal sheet φ_h (i.e. of the locus of the focal points F_h as $\mathring{\mathbf{p}}$ moves along C_2)

$$\begin{gathered} x_1=\lambda_h d_{24}+\mu_h d_{23}, \quad x_2=\lambda_h d_{41}+\mu_h d_{31}, \\ x_3=\mu_h d_{12}, \quad x_4=2\lambda_h d_{12}, \\ \tfrac{1}{2} d_{ij}=2\left(\frac{\partial}{\partial\eta^{\mathrm{I}}} p_{[i}\right)\left(\frac{\partial}{\partial\eta^{\mathrm{II}}} p_{j]}\right), \end{gathered} \tag{1}$$

where λ_h, μ_h are related by a quadratic equation. A detailed discussion of (1) leads the author to the statement: If φ_h is a surface, then its tangent plane in F_h is f_h.
V. Hlavatý (Bloomington, Ind.).

Aleksandrov, A. D. Ruled surfaces in metric spaces. Vestnik Leningrad. Univ. **12** (1957), no. 1, 5–26, 207. (Russian. English summary)

Svoboda, Karel Sur une caractérisation métrique de la surface de Véronèse. Publ. Fac. Sci. Univ. Masaryk **1955**, 407–428. (Russian summary)

L'Autore si occupa delle superficie (V) di uno spazio a 5 dimensioni, S_5, a curvatura costante c $(>0\leqq 0)$, tali che, in ogni punto P, l'indicatrice della curvatura normale di (V) sia una circonferenza di raggio costante r, situata nello S_3 ortogonale al piano tangente a (V) in P, e il cui centro ha una distanza $v\neq 0$ da P. [Per il caso $v=0$ cf. O. Borůvka, Publ. Fac. Sci. Univ. Masaryk, no. **106** (1929)]. L'Autore trova: i) che v è necessariamente costante; ii) che ogni superficie (V) è necessariamente una superficie di Veronese, di cui si dànno interessanti caratterizzazioni geometriche, in relazione all'assoluto dello spazio S_5. I risultati sono ottenuti impiegando il metodo del "repère mobile" di E. Cartan. *V. Dalla Volta.*

Ryžkov, V. V. On the order of applicability of surfaces with corresponding conjugate systems. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 338–340. (Russian)

Blaschke, Wilhelm. Über die Differentialgeometrie besonderer Gruppen. Schr. Forschungsinst. Math. **1** (1957), 131–137.

Paper begins with a review of some problems of affine, conformal, and projective differential geometry. Most of it is devoted to an exposition of the "geometry of webs" [Cf. Blaschke und G. Bol, Geometrie der Gewebe, Berlin 1938]. The study of hexagonal and octahedral surface webs whose surfaces are planes is a problem intimately related to abelian integrals on algebraic curves.

S. Chern (Chicago, Ill.).

Ostrowski, Alexander. Über die Verbindbarkeit von Linien- und Krümmungselementen durch monoton gekrümmte Kurvenbögen. Enseignement Math. (2) **2** (1956), 277–292.

The class Γ consists of the oriented arcs in a euclidean plane with a positive continuous monotonically nonincreasing radius of curvature which is smaller at the endpoint than at the initial point and constant only in a finite number of intervals; their total curvature is less than 2π. Theorem I: Given two oriented line elements (A, α) and (B, β) such that the directions of the lines α and β are separated by AB. Then there is an arc in Γ with the initial point A, the initial tangent α and the corresponding end-elements B and β if and only if

(1) $\sphericalangle(\alpha, \mathbf{AB}) < \sphericalangle(\mathbf{AB}, \beta)$.

The necessity of this condition was proved by Vogt and others under stronger assumptions [cf. Hirano, Tôhoku Math. J. **47** (1940), 126–128; MR **2**, 261, and the references in the review]. The sufficiency follows from Theorem II: Suppose (1) and the assumptions of Theorem I are satisfied. Let a $[b]$ denote a point on the normal of α $[\beta]$ at A $[B]$ such that a and B $[b$ and $A]$ lie on the same side of α $[\beta]$. Then there is an arc of class Γ with the initial point A, the initial center of curvature a and the corresponding end-elements B and b if and only if $|aA|>|ab|+|bB|$ [i.e. if the circle through A about a contains that about b through B. The necessity of this condition was proved by Haller [S.-B. Phys.-Med. Soz. Erlangen **69** (1937), 215–218], — and possibly earlier by Hjelmsler — under somewhat weaker assumptions. *P. Scherk.*

★ **Efimow, N. W. Flächenverbiegung im Grossen. Mit einem Nachtrag von E. Rembs und K. P. Grotemeyer.** Akademie-Verlag, Berlin, 1957. xi+233 pp. DM 33.50.

A translation from the Russian of the article reviewed in MR **10**, 324.

★ **де Рам. Ж. [de Ram, Ž. (de Rham, G.)] Дифференцируемые многообразия. [Variétés différentiables.]** Izdat. Inostr. Lit., Moscow 1956. 250 pp. 10.20 rubles.

A translation by D. A. Vasil'kov (with a preface by P. S. Alexandrov) of the book reviewed in MR **16**, 957.

Schouten, J. A. On currents and their invariant derivatives. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 371–380, 381–385.

Currents in the sense of de Rham are linear functionals on differential forms (or W-forms) over a manifold M, with certain continuity properties. On the one hand they generalize differential forms, because if ω is any p-form and α a fixed $(n-p)$-form, then $T[\omega]=\int_M \alpha\wedge\omega$ defines a current. (Of course, one has to make some assumptions like (a) M is orientable; or either α or ω is a W-form; (b) M is compact; or either α or ω has a compact carrier; or variations of these.) On the other hand, currents are generalizations of differentiable chains, because if c^p is a p-chain, then $T[\omega]=\int_{c^p}\omega$ is a current. The author also considers currents of the types $T=(c^{p+q}, \alpha_q)$, $T=(c^{p-q}, \beta_q)$, where

$$(c^{p+q},\ \alpha_q)[\omega_p]=\int_{c^{p+q}} \alpha_q\wedge\omega_p;$$

$$(c^{p-q},\ \beta^q)[\omega_p]=\int_{c^{p-q}} \beta^q\cdot\omega_p\quad (q\leqq p),$$

where

$$\beta^q\cdot\omega_p=\beta^{\lambda_1\cdots\lambda_q}\omega_{\lambda_1\ldots\lambda_p}\,dx^{\lambda_{q+1}}\wedge\cdots\wedge dx^{\lambda_p}.$$

Finally, the most general case considered is

$$(c^{p-q+r},\ \beta_r{}^q)[\omega_p]=\int_{c^{p-q+r}} \beta_r{}^q\cdot\omega_p\quad (q\leqq p),$$

where

$$\beta_r{}^q\cdot\omega_p=\beta^{\lambda_1\cdots\lambda_q}_{\mu_1\cdots\mu_r}\,\omega_{\lambda_1\cdots\lambda_p}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r}\wedge dx^{\lambda_{q+1}}\wedge\cdots\wedge dx^{\lambda_p}.$$

Besides the exterior derivative $d\omega$ of a differential form ω there is also $\underset{\beta}{\pounds}{}_r{}^{q-1}\omega_p$, defined by

$$(\underset{\beta}{\pounds}{}_r{}^{q-1}\omega_p)_{\lambda_1\cdots\lambda_{r+p-q+1}}=q\beta^{\kappa_1\cdots\kappa_q}_{[\lambda_1\cdots\lambda_r}\partial_{|\kappa_1}\omega_{\kappa_2\cdots\kappa_q|\lambda_{r+1}\cdots\lambda_{r+p-q+1}]}+$$
$$(-1)^{q+1}(p-q+1)\omega_{\kappa_1\cdots\kappa_q[\lambda_{r+2}\cdots\lambda_{r+p-q+1}}\partial_{\lambda_{r+1}}\beta^{\kappa_1\cdots\kappa_q}_{\lambda_1\cdots\lambda_r]}.$$

The operator $\underset{\beta}{\pounds}{}_1{}^0$ is the exterior derivative d; $\underset{v}{\pounds}{}_0{}^0$ is the Lie derivative with respect to the vector field $v=v_0{}^1$; $\underset{L}{\pounds}{}_p{}^0\omega$ is the same as $[L, \omega]$ (apart from a trivial numerical factor) in the paper of Frölicher and Nijenhuis [same Proc. **59** (1956), 338–359; MR **18**, 569].

For the case $T=(M, \alpha)$ one has $dT[\omega_p]=(-1)^{n-p}T[d\omega_p]$ and this is extended to other currents by definition. Then it follows that

$$d(c^{n-q}, \alpha_p)=(-1)^{q-1}(bc^{n-q}, \alpha_p)+(-1)^q(c^{n-q}, d\alpha_p),$$
$$(n-p)d(c^{n-q}, \beta^{q-p})=(-1)^{q-1}(n-q)(bc^{n-q}, \beta^{q-p})+$$
$$(-1)^{p+1}(c^{n-q}, \underset{\beta}{\pounds}{}^{q-p-1}),$$

where b is the boundary operator. The most general case is

$$(n-p+q-r)d(c^{n-q}, \beta_{r-q}{}^{q-p})=$$
$$(-1)^{r+1}(n-r)(bc^{n-q}, \beta_{r-q}{}^{q-p})+$$
$$(-1)^{p-q+r+1}(c^{n-q}, \underset{\beta}{\pounds}{}_{r-q}{}^{q-p-1}).$$

The operator $\underset{\beta}{\pounds}$ can also be defined for currents, by observing what one gets in the case $T=(M, \alpha)$. While for $\beta=v=$vector field one has $\underset{\beta}{\pounds}T[\omega]=-T[\underset{\beta}{\pounds}\omega]$, the formula for $\underset{\beta}{\pounds}\iota^{k-1}T$ on currents gives rise to another currency symbol: $\underset{\beta}{\pounds}\iota^{k-1}T[\omega]=-T[\underset{\beta}{\$}\iota^{k-1}\omega]$, where $\underset{\beta}{\$}$ is a sum of $\underset{\gamma}{\pounds}$'s with various γ's obtained from β by contractions, each with a suitable coefficient. If β and γ are purely contravariant, the commutator $\underset{\beta}{\pounds}^p\underset{\gamma}{\pounds}^q+(-1)^{pq}\underset{\gamma}{\pounds}^q\underset{\beta}{\pounds}^p$ is, within a numerical factor, precisely $\underset{[\beta,\gamma]}{\pounds}^{p+q}$, where $[\beta, \gamma]$ is the Schouten differential concomitant [same Proc. **43** (1940), 449–452; MR **2**, 200].

In order to facilitate the computations, the author introduces the "valence-number" notation, which allows a substantial reduction of the number of indices to be written down. Using this method he succeeds in deducing a formula for $\underset{\beta}{\pounds}\iota^{k-1}(\omega\wedge\pi)$. *A. Nijenhuis.*

Schouten, J. A. On currents and their invariant derivatives. III. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 1–11.

This part III deals with a further development of the valence-number method introduced in part II reviewed above, and with the fixing of suitable constants in an algorithm of alternating (co- and contravariant) tensors, so as to simplify the algorithm. Two sets of constants are proposed: those arising from the standard alternation formulas of Ricci calculus as developed by the author in his work; and those arising from generalizations of E. Cartan's methods. Applications to the original problem of this sequence of papers — operations on currents — are postponed to a following part.

If $P_m{}^h$, $Q_q{}^k$ are alternating tensors with h (k) contravariant and m (q) contravariant indices, the author defines $[P_m{}^h \overset{s}{,} Q_q{}^k]$ as a set of components which involve the components of P and Q and their first order partial derivatives. It is alternating in its $h+k-s$ upper and $q+m-s+1$ lower indices. In general, it is not a tensor; but for four special sets of values of h, k, q, m, s it is one. I. $h=k=s=1$; then P and Q are vector forms in the sense of Frölicher and Nijenhuis [see reference in review above], and $[P_m{}^1 \overset{s}{,} Q_q{}^1]$ is just their differential concomitant $[P, Q]$, which is also a vector form. II. $s=h=1$, $m=0$; then $[P_m{}^1 \overset{s}{,} Q_q{}^1]$ is the Lie derivative of the tensor field Q with respect to the vector field P. III. $s=1$, $q=m=0$; then $[P^{h1} \overset{1}{,} Q^k]$ is the Schouten differential concomitant of the alternating contravariant tensor fields P and Q [ibid. **43** (1940), 449–452; MR **2**, 200]. IV. $h=s$, $k=0$; then $[P_m{}^s \overset{s}{,} Q_q]$ is reducible to the form

$$P \wedge_s dQ+(-1)^{m+s+1}d(P \wedge_s Q);$$

where $\wedge_s$ is an algebraic operator generalizing the exterior product symbol $\wedge$ for differential forms, and the operation $\bar{\wedge}$ for vector forms. *A. Nijenhuis.*

Schouten, J. A. On currents and their invariant derivatives. IV. Nederl. Akad. Wetensch. Proc. Ser. A. **60** = Indag. Math. **19** (1957), 233–241.

If $P_p{}^h$ and $Q_q{}^l$ are alternating tensors, and $s\leqq h$, q, then $P\wedge_s Q$ is the tensor with components

$$\gamma P^{\kappa_1\cdots\kappa_s[\kappa_{s+1}\cdots\kappa_h}_{[\mu_1\cdots\mu_p}\, Q^{\lambda_1\cdots\lambda_l]}_{|\kappa_1\cdots\kappa_s|\nu_{s+1}\cdots\nu_q]}$$

where γ is suitably chosen [in either one of the ways fixed in Part III, reviewed above With this notation,

$$\underset{P}{\pounds}\omega=[P_p{}^h, \omega_q]=P\wedge_h d\omega+(-1)^{p+h+1}d(P\wedge_h\omega).$$

The symbol $_{|s}$ is defined by $P_p{}^h{}_{|s}$ having components $\gamma P^{\lambda_1\cdots\lambda_s\lambda_{s+1}\cdots\lambda_h}_{\lambda_1\cdots\lambda_s\mu_{s+1}\cdots\mu_p}$, with suitable γ. Fundamental identities for $\wedge_s$, $_{|s}$ and $\pounds$ are derived, such as 'associative' rules for $\wedge_s$, 'distributive' rules for $\wedge_s$ and $_{|s}$, product rules for $\underset{P}{\pounds}(\varphi\wedge\psi)$. The original problem — operations on currents — is taken up again, and operations like $P_q{}^r\wedge_r T_p$, $\underset{P}{\pounds}T$ are defined and studied, where T is a current. The present symbolic method turns out to be sufficiently more effective than the straightforward components-only method of Parts I, II that it is now possible to compute explicitly $\underset{P}{\pounds}T[\varphi]=\sum_{u=0}^{k}(-1)^{\alpha}T[\underset{P_{|u}}{\pounds}\varphi]$, where α is a polynomial in u, the degrees of $P=P_l{}^k$, $\varphi=\varphi_p$, and the dimension of the space. *A. Nijenhuis* (Seattle, Wash.).

Barner, Martin. Über die Mindestanzahl stationärer Schmiegebenen bei geschlossenen streng-konvexen Raumkurven. Abh. Math. Sem. Univ. Hamburg **20** (1956), 196–215.

A closed curve C of class $C^{(n)}$ in a projective space R_n of dimension n is said to be strongly convex if through any $n-1$ points of C there is a hyperplane having no other intersections with C. The purpose of this paper is to establish the theorem that in R_n if a hyperplane has k points in common with a strongly convex curve C, then C has at least k stationary osculating hyperplanes. The author proves this theorem first for $n=2$ and then for the general case by using induction on dimension. This theorem has various interesting applications. For instance, from this theorem the four-vertex theorem and the theorem concerning the six sextactic points of an oval can be deduced as special cases, and the four-vertex theorem can be extended to convex curves in an Euclidean space E_{2m} of even dimension $2m$ by means of the following definitions. (i) A curve C in E_{2m} is convex if a hyperplane (or hypersphere) passing through any $2m$ points of C has no other intersections with C. (ii) A vertex of a curve C in E_{2m} is a point at which the osculating hypersphere of C is stationary. *C. C. Hsiung* (Bethlehem, Pa.).

Mihăilescu, Tiberiu. Sur le repère normal projectif d'une surface. Rev. Math. Pures Appl. **1** (1956), no. 2, 107–132.

Let A_0, A_1, A_2, A_3 be the vertices of the normal tetrahedron of Wilczynski associated with a moving regular point A_0 of a nonruled surface S_0 in a three-dimensional projective space, and let S_i ($i=1, 2, 3$) be the surfaces described by the vertex A_i as A_0 moves over S_0. The purpose of this paper is to obtain some symmetric relations among the surfaces S_0, S_1, S_2, S_3. Let Σ_1, Σ_2 be respectively the envelopes of the planes $A_1A_2A_3$, $A_0A_2A_3$ as the point A_0 moves over S_0; and let B_1, B_2 be the points of contact of the planes $A_0A_2A_3$, $A_0A_1A_3$ with the surfaces Σ_1, Σ_2 respectively. The surface S_0 is isothermally asymptotic if and only if the lines A_1B_1 and A_2B_2 are coincident. If the two focal points of the line A_1A_3 are harmonic conjugate with respect to the vertices A_1 and A_3, then the focal curves of the congruence A_1A_3 form a conjugate net on S_0. The determination of the class of surfaces on which the focal curves of the congruence A_1A_3 form a conjugate net is given, together with similar discussions on the cases in which S_1, Σ_2, S_0 or S_1, S_2, Σ_1, Σ_2 are in an asymptotic correspondence. *C. C. Hsiung.*

Gheorghiu, Gh. Th. Les courbes Tzitzéica dans la géométrie projective. Rev. Math. Pures Appl. **1** (1956), no. 2, 133–150.

A curve C in an ordinary Euclidean space is called a

curve of Tzitzéica of the first kind if Td^2 is constant, where T is the torsion of C at every point P, and d the distance from a fixed point in the space to the osculating plane of C at P. Using covariant differentiation and the calculus of G. Sannia it is shown that each of the following two conditions is necessary and sufficient for a curve C to be a curve (T) of Tzitzéica of the first kind: (i) The normal coordinates of every point of C are also Cartesian coordinates; and (ii) the projective torsion at every point of C vanishes. Moreover, the vertex y_1 of the coordinate tetrahedron $yy_1y_2y_3$ of Sannia associated with a moving point y on a curve (T) describes a plane curve. Finally, at every point of a curve (T) the projective rectifying plane coincides with the affine rectifying plane and passes through a fixed point; and the projective arc is proportional to the affine arc. *C. C. Hsiung* (Bethlehem, Pa.).

Villa, Mario. Applicabilità proiettiva fra superficie di 2a specie della V_4 di Segre. Boll. Un. Mat. Ital. (3) **11** (1956), 493–495.

A surface on a four-dimensional variety V_4 of Segre is said to be of the second kind if two of its three ∞^1-systems of quasi-asymptotic curves coincide. In this paper the notion [see M. Villa and L. Muracchini, same Boll. (3) **10** (1955), 313–327; MR **17**, 781] of projective applicability between two surfaces on V_4 with three distinct ∞^1-systems of quasi-asymptotic curves is extended to two surfaces Σ and Σ' of the second kind; and between Σ and Σ' two special projective applicabilities, called strong and inverse projective applicabilities respectively, are defined.

C. C. Hsiung (Bethlehem, Pa.).

Speranza, Francesco. Applicabilità proiettiva fra trasformazioni puntuali di 2a specie. Boll. Un. Mat. Ital. (3) **11** (1956), 526–537.

In this paper, the author expresses the projective applicabilities between two surfaces Σ and Σ' of the second kind defined in the paper reviewed above in terms of two point-transformations T and T' of the second kind between two pairs of planes; and thus defines, between T and T', the corresponding projective applicabilities, for each of which analytic conditions are obtained.

C. C. Hsiung (Bethlehem, Pa.).

Takasu, Tsurusaburo. Non-holonomic Laguerre fibre bundle geometry. Yokohama Math. J. **4** (1956), 1–46.

Im Raum der $(n-1)$-Sphären im R^n, vom Verf. n-dim. "Laguerre Raum" L^n genannt, lassen sich ausgezeichnete "Laguerre Koordinaten" einführen, indem jeder $(n-1)$-Sphäre des R^n mit dem Mittelpunkt $(x_1, \cdots, x_n)$ und dem Radius r der Punkt $(x_1, \cdots, x_n, r) \in R^{n+1}$ zugeordnet wird. Die Untergruppe der $(n+1)$-dim. allgemeinen linearen Gruppe, welche die quadratische Form

$$y_1^2+\cdots+y_n^2-y_{n+1}^2$$

invariant läßt, wirkt deshalb auf L^n als "Laguerre Gruppe" und läßt als solche den tangentialen Abstand zweier $(n-1)$-Sphären invariant. — Verf. betrachtet verschiedene lineare Übertragungen in einer $(n+1)$-dim. Mannigfaltigkeit M mit einer (nicht-riemannschen) Metrik, wie sie durch die obige quadratische Form charakterisiert ist. Viele bekannte Sätze über lineare Übertragungen in solchen Mannigfaltigkeiten — und nur solche Sätze finden sich in dieser Arbeit — können somit als Sätze der Geometrie eines Faserraumes mit M als Basis, L^n als Faser und der "Laguerre Gruppe" als Strukturgruppe interpretiert werden; außerdem werden (z.B. im Hinblick auf spezielle Darstellungen der Übertragungsparameter) einige besondere m-Bein-Felder betrachtet ("Veblen frame", "natural frame"). Im übrigen gilt für die Darstellung und für die in dieser Arbeit benutzten Begriffe "$\cdots$ in the large," "holonomic" und "non-holonomic geometry" entsprechendes wie für einige frühere Arbeiten des gleichen Verf. [dasselbe J. **1** (1953), 1–28, 29–38, 39–74, 75–77, 79–82, 83–87; MR **15**, 350].

P. Dombrowski und F. Hirzebruch (Bonn).

See also: Craig, p. 43.

Manifolds, Connections

Vranceanu, Georges. Les transformations crémoniennes entières et les espaces à connexion affine. C. R. Acad. Sci. Paris **243** (1956), 1997–1999.

To an entire Cremona transformation $u^i=u^i(x^1, x^2, \cdots, x^n)$ $(i=1, \cdots, n)$ corresponds a locally euclidean affine space A_n determined by the equations $\partial^2 u/\partial x^j\partial x^k=-\Gamma_{jk}{}^s\partial u/\partial x^s$ which have the u^i as solutions. The A_n is globally equivalent to the uclidean space $E_n(u^1, \cdots, u^n)$. The converse is also true if the $\Gamma_{jk}{}^i$ are constants. These entire Cremona transformations are for $n\leqq 2$ all of the form of Jonquières $u^1=x^1$, $u^2=x^2+\varphi(x^1)$, φ a polynomial; for $n\geqq 3$ some entire Cremona transformations are presented which are not of the Jonquières type; to these transformations the corresponding $\Gamma_{jk}{}^s$ are given. [Reference is made to W. Engel, Math. Ann. **130** (1955), 11–19; MR **17**, 787; B. Segre, Communication faite au 3e Congrès Math. Soviet, Moscou, juin, 1956; and H. W. E. Jung, J. Reine Angew. Math. **184** (1942), 161–174; MR **5**, 74.]

D. J. Struik (Cambridge, Mass.).

Teleman, C. Sur les espaces à connexion affine A_1 complexes. Rev. Math. Pures Appl. **1** (1956), no. 2, 151–165.

Let A_2 be a two-dimensional space with affine connection, x^1, x^2 a coordinate system, and $\Gamma_{jk}{}^i(x^1, x^2)$ $(i, j, k=1, 2)$ the coefficients of the connection in a general neighbourhood V of A_2. The A_2 is called a K_2 if the following conditions are satisfied. α) The equations of parallel displacement in V, $dv^j=\Gamma_{rs}{}^jv^rdx^s$, can be written in the form $dw=\Gamma wdz$, where $w=v^1+iv^2$ and $z=x^1+ix^2$, and "the connection" Γ is a holomorphic function of z in V. β) In any non-void intersection of two neighbourhoods $V(z)$ and $V'(z')$ we have $z'=\varphi(z)$, where φ is holomorphic and univalent in $V\cdot V'$. γ) The connections $\Gamma(z)$ in $V(z)$ and $\Gamma'(z')$ in $V'(z')$ satisfy the transformation formula

$$\frac{d^2z'}{dz^2}=\Gamma'\left(\frac{dz'}{dz}\right)^2-\Gamma\frac{dz'}{dz}. \tag{1}$$

A model of a space K_2 is a Riemann surface. It can be regarded as a complex A_1. It is also locally euclidean since, because of α), the parallel transport does not depend upon the transport curve. In every neighbourhood it is therefore possible to reduce the connection to zero by a suitable change of the variable z, using formula (1). Inversely one can associate to every analytic function a space K_2.

The author defines, in continuation, two spaces which can be derived from a given K_2: 1) A $\tilde{K}$ is the universal covering surface of a K_2; 2) a K_2^* is obtained, roughly, from a given K_2 by an analytic change of the coordinate z, the corresponding connections being related by formula (1).

A number of results are obtained by the author concerning these spaces, of which the following is an example: The poles of the connection Γ of K_2^* are all simple with rational residues $\neq 1$. *R. Blum* (Saskatoon, Sask.).

Moór, Arthur. Allgemeine metrische Räume von skalarer Krümmung. Publ. Math. Debrecen **4** (1956), 207–228.

The spaces considered by the author are spaces where fundamental tensors depend not only upon position coordinates but also on the components of a contravariant or covariant vector density of arbitrary weight p. They are therefore generalizations of Finsler spaces (contravariant case with $p=0$) and of Cartan spaces (covariant case with $p=-1$). The paper contains generalizations of some results given for Finsler spaces by Berwald [Ann. of Math. (2) **42** (1941), 84–112; **48** (1947), 755–781; MR **2**, 304; **9**, 207]. The chief of these concerns Berwald's definition of curvature for the bidirection determined by the fundamental unit vector l and an arbitrary direction η, and denoted by $B(x, u, \eta)$, where u is the contravariant or covariant "element of support." The author points out that in the general case two generalizations of Berwald's notion are possible. Various conditions are given for the coincidence of these two generalizations. Some more details are given for the case of a two-dimensional space. Spaces of scalar curvature are those for which $B(x, u, \eta)$ is independent of η. The author concludes the paper with some special results valid for such spaces. *E. T. Davies* (Southampton).

See also: Chen, p. 12; Libois, p. 103; de Mira Fernandes, p. 108.

Complex Manifolds

★ **Calabi, Eugenio. On Kähler manifolds with vanishing canonical class.** Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 78–89. Princeton University Press, Princeton, N. J., 1957. $7.50.

Soit M une variété kählérienne compacte, et $R_{\alpha\beta^*}$ le tenseur de Ricci; $\Sigma = iR_{\alpha\beta^*}dz^\alpha \wedge d\bar{z}^\beta$ est appelée forme de Ricci de la métrique; la classe de cohomologie de $(\frac{1}{2}\pi)\Sigma$ ne dépend que de la structure analytique complexe de M et est appelée classe canonique (ou première classe de Chern) de M. La forme extérieure associée à la métrique sera dite principale. Soit A_M la variété d'Albanese de M et, un couple de points homologues ayant été choisi, soit J l'application canonique de M dans A_M. Proposition: Supposons la métrique de M indéfiniment différentiable, de forme principale ω et de forme de Ricci Σ. Si Σ' est une (1, 1)-forme C^∞, réelle, fermée et cohomologue à Σ, alors, il existe une métrique kählérienne unique C^∞, de forme principale ω' cohomologue à ω et ayant Σ' pour forme de Ricci. La proposition est rendue plausible par un raisonnement heuristique. On suppose maintenant que la classe canonique de M est nulle, alors, M est dite kählérienne spéciale. Théorème 1: Si M a la dimension complexe n et l'irrégularité g_1 (premier nombre de Betti $2g_1$), on a $0 \leq g_1 \leq n$; de plus: J est une surjection; M est un espace fibré analytique complexe de base A_M, de projection J, de groupe structural fini et abélien, de fibre kählérienne spéciale et connexe. La démonstration repose sur l'existence (d'après la Proposition) d'une métrique kählérienne sur M dont la forme de Ricci est nulle. Le theorème 1 peut être interprété ainsi: (Théorème 2) Il existe un revêtement fini $\tilde{M}$ de M qui est le produit direct de sa propre variété d'Albanese et d'une variété kählérienne spéciale régulière (i.e. telle que $g_1=0$) qui peut être réduite à un point; le groupe du revêtement $\tilde{M}$ de M est résoluble. On en déduit une classification des variétés kählériennes spéciales en fonction de celles qui sont régulières. Corollaire: Le m-genre de M étant la dimension de l'espace vectoriel des densités holomorphes de poids m sur M, supposons que le revêtement $\tilde{M}$ de M ait k feuillets, alors, il existe un diviseur d de k tel que le m-genre de M soit 1 ou 0 suivant que d divise m ou non. *P. Dolbeault.*

★ **Kodaira, Kunihiko. Some results in the transcendental theory of algebraic varieties.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 474–480. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Cet article contient un résumé de résultats obtenus par l'auteur seul ou en collaboration avec D. C. Spencer, jusqu'en 1954, sur les faisceaux analytiques des variétés analytiques complexes, avec application à la géométrie algébrique; la technique fait intervenir la théorie du potentiel. Les sujets traités sont les suivants: espaces fibrés analytiques complexes à fibre vectorielle de dimension un [Kodaira et Spencer, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 868–872; MR **16**, 75]; cohomologie à coefficients dans certains faisceaux analytiques [Kodaira; ibid. **39** (1953), 865–868, 1268–1273; MR **16**, 74, 618]; caractérisation des variétés algébriques projectives sans singularité comme variétés de Hodge [Kodaira, Ann. of Math. (2) **60** (1954), 28–48; MR **16**, 952]; variétés de Picard [Kodaira et Spencer, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 868–872, 872–877; MR **16**, 75]; défaut caractéristique d'un diviseur premier non singulier [Kodaira and Spencer, ibid. **39** (1953), 1273–1278; MR **16**, 857]. L'article se termine par une application inédite des résultats précédents: Soient V une variété algébrique projective sans singularité, K l'espace fibré canonique sur V, S un diviseur premier non singulier de V tel que l'espace fibré $\{S\}-K$ soit positif. Alors, le système continu complet $\mathscr{C}$ contenant S est déterminé de façon unique et est formé de tous les diviseurs effectifs homologues à S sur V. Le système $\mathscr{C}$ peut être paramétré par une variété algébrique non singulière qui est un espace fibré analytique sur la variété de Picard attachée à V et dont la fibre est un espace projectif. Ainsi $\mathscr{C}$ est formé de ∞^q systèmes linéaires, q désignant l'irrégularité de V. De plus, le système linéaire caractéristique de $\mathscr{C}$ sur S est complet. *P. Dolbeault* (Bordeaux).

Frölicher, Alfred; and Nijenhuis, Albert. Some new cohomology invariants for complex manifolds. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 540–552, 553–564.

In an earlier paper [same Proc. **59** (1956), 338–359; MR **18**, 569], the authors developed the theory of vector-valued differential forms (briefly vector forms) on a differentiable manifold. The usual bracket operation on vector fields was generalized: (i) to a bracket operation $[M, \omega]$ between a vector form M and a scalar (i.e. ordinary) form ω; (ii) to a bracket operation $[M, N]$ between two vector forms M, N.

In Part I of the present paper the theory of vector forms is applied to almost complex manifolds. In particular it is shown that if J is an almost complex structure on a differentiable manifold X (i.e. an automorphism of the tangent bundle of X with $J^2=-I$, where I is the

identity), then $[J, J]=0$ is the usual integrability condition which, when X is real analytic, is the condition that J should define a complex structure on X. The equivalence of several different forms of the integrability condition also follows easily from the machinery of vector forms.

In Part II the authors study the cohomology groups of a complex manifold X with coefficients in the sheaf of germs of holomorphic forms on X. If the forms are scalar forms these are the Dolbeault cohomology groups. If the forms are vector forms these are the GLA-cohomology groups (authors' notation). The bracket operation (ii) above gives rise to a graded Lie algebra structure for the GLA-cohomology. The bracket operation (i) above gives rise to a representation of the GLA-cohomology on the Dolbeault. This representation is trivial if X is a compact Kähler manifold. *M. F. Atiyah.*

See also: Gamkrelidze, p. 53.

Algebraic Geometry

Manara, Carlo Felice. Idee classiche ed idee moderne sulla geometria algebrica. Period. Mat. (4) **35** (1957), 1–13.

★**Gröbner, W. Über die idealtheoretische Grundlegung der algebraischen Geometrie.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 447–456. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

In der vorliegenden Arbeit wird ein kurzer Überblick über die vom Verfasser bevorzugte idealtheoretische Grundlegung der algebraischen Geometrie gegeben, die in seinem Buch "Moderne algebraische Geometrie" [Springer, Wien-Innsbruck, 1949; MR 11, 536] näher ausgeführt ist. Verfasser verteidigt hierbei seinen mehrfach angegriffenen Standpunkt, für den vor allem die Strenge, Einfachheit und methodische Geschlossenheit spricht. Dies gilt auch besonders für den Multiplizitätsbegriff, der bekanntlich mit dem üblichen, durch Stetigkeitsbetrachtungen gewonnenen nicht immer übereinstimmt, aber doch nach Ansicht des Verfassers die erwähnten Vorteile hat. *W. Burau* (Hamburg).

★**Northcott, D. G. Specialization methods in algebraic geometry.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 489–492. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Analyzing the solutions of the extension problem for specializations over a field, the author is led to an equivalent formulation in terms of ring homomorphisms. The definition is the following: Let Q be a local domain, $\mathfrak{m}$ its maximal ideal and $K=Q/\mathfrak{m}$. Let now $(y_1, \cdots, y_s)$ be a set of elements of a field containing Q and $(\beta_1, \cdots, \beta_s)$ a set of elements belonging to an extension field of K. Then one says that (β) is a specialization of (y) with respect to Q if the two conditions, (i) $\varphi(Y_1, \cdots, Y_s)$ is a polynomial with coefficients in Q and (ii) $\varphi(y)=0$, together imply that $\bar{\varphi}(\beta)=0$, where $\bar{\varphi}(Y)$ is the polynomial obtained from $\varphi(Y)$ by reading it modulo $\mathfrak{m}$.

To show the usefulness of this definition the author gives a pleasant proof of Zariski's Main Theorem on birational correspondences. [See also Northcott, Proc. London Math. Soc. (3) **1** (1951), 129–137; MR 13, 867.] *E. Lluis* (Mexico, D.F.).

Chisini, Oscar. Sul principio di continuità. Period. Mat. (4) **34** (1956), 265–277.

An expository lecture on the principle of continuity in algebraic geometry, beginning with the ideas of Kepler.

Bertoldi, I. "Enumeratio linearum tertii ordinis" di J. Newton. Period. Mat. (4) **35** (1957), 14–43.

An expository discussion with diagrams.

Godeaux, Lucien. Remarques sur la formation des systèmes canonique et pluricanoniques de quelques surfaces algébriques. III, IV. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 8–16, 56–62.

Parts I and II are listed in MR **18**, 415, 935.

Godeaux, Lucien. Sur les surfaces cubiques non réglées osculant un plan le long d'une droite. Mathesis **66** (1957), 5–7.

Détermination, en utilisant les propriétés de la polarité, des singularités d'une surface cubique non réglée assujettie à osculer un plan le long d'une droite.
Résumé de l'auteur.

Godeaux, Lucien. La théorie des involutions cycliques appartenant à une surface algébrique et ses applications. Bull. Soc. Roy. Sci. Liège **26** (1957), 3–15.

Facciotti, Guido. Sui fasci di quadriche. Period. Mat. (4) **34** (1956), 284–293.

Historical and expository remarks about pencils of quadric surfaces and corresponding twisted cubic curves.

Abhyankar, Shreeram. On the field of definition of a nonsingular birational transform of an algebraic surface. Ann. of Math. (2) **65** (1957), 268–281.

The author proves the following theorem: Given an arbitrary two-dimensional algebraic function field K/k, a finite algebraic purely inseparable extension k' of k can be found such that, if K' is a free join of k' and K over k, then nonsingular projective models of K'/k exist which are absolutely nonsingular over k'. If k is perfect, $K'=K$. The proof consists of extending a theorem of local uniformization on algebraic surfaces over algebraically closed modular ground fields, proved by the author in a former paper, to a two-dimensional algebraic function field over a perfect ground field, from which the above final result is derived. Some further problems are suggested in the last section. *T. R. Hollcroft* (Aurora, N.Y.).

Baldassarri, Mario. Una caratterizzazione delle varietà abeliane e pseudo-abeliane. Ann. Mat. Pura Appl. (4) **42** (1956), 227–252.

This paper, which seeks to identify abelian and pseudo-abelian varieties with the class of varieties for which certain canonical systems are of order zero, is unfortunately vitiated by an error. The basic lemma (Theorem 6, p. 243), which asserts that a non-singular variety on which the canonical series has order zero is irregular, is false; a counter-example being the V_3 obtained from S_3 by dilatation of a non-singular curve of genus 3.

The author's approach to the problem is interesting, and it is conceivable that his methods could be used to characterise the varieties under discussion, though not, naturally, in the form stated in this paper. *J. A. Todd.*

See also: Lech, p. 11; Kodaira, p. 62.

NUMERICAL ANALYSIS

Numerical Methods

★ **Van Laethem, Marcel. Une méthode nouvelle et générale de calcul des intégrales généralisées. Théorie et pratique à l'usage des mathématiciens, physiciens et ingénieurs.** Editions Nauwelaerts, Louvain, 1956. viii+180 pp. (3 plates) 250 francs belges.

The book owes its existence to the problem of evaluating indefinite integrals which accompany the calculation of the trajectory of an ion through the field of an axially symmetric condenser. The main problem is related to an integral of the type $\psi=\int_b^x y\,dx$, $y=(x-b)^n\cdot f(x)$, $f(b)\neq 0$, $f(x)$ continuous, $n>-1$. Especially the case $n=-\frac{1}{2}$ is treated with many details. Here the author assumes that x admits an expansion in powers of ψ; the coefficients of this expansion can be calculated from the function $\iota=y'y^{-3}$ and its derivatives. Finally the expansion for x is inverted so as to obtain an expansion of ψ^2 in powers of x. Formulas for the coefficients are listed up to the order 18. In the case $n<-\frac{1}{2}$ it is still suggested to expand x in powers of ψ; in the case $n>-\frac{1}{2}$ the author recommends to expand ψ in powers of y^{-1}: $\psi=\sum_0^\infty R_k y^{-k}$. {It is not clear to this reviewer how the case $b=0$, $y=x^{-2/5}$ would have to be treated in this way; obviously the function ψ cannot be expanded in powers of y^{-1}.} Several numerical applications of the proposed methods are presented. They refer to the case $y=(1-x^2)^{-\frac{1}{2}}$, to elliptic integrals, and to the above mentioned trajectory of an ion. The new method is compared with already known ones. It appears that the results of the application to the trajectory of an ion exceed the boundaries of numerical analysis, since physical aspects of the problem are discussed in detail. Finally the author indicates how his methods can be extended to the case of integrals over infinite intervals. *H. Bückner* (Schenectady, N.Y.).

Hitchcock, A. J. M. Polynomial approximations to Bessel functions of order zero and one and to related functions. Math. Tables Aids Comput. **11** (1957), 86–88.

The author gives approximation formulas for the Bessel functions of the first kind $J_0(x)$, $J_1(x)$ and of the second kind $Y_0(x)$, $Y_1(x)$, and for the integral $\bar{J}_0(x)=\int_0^x J_0(t)dt$, valid for all values of x. Approximations of the same type are also given for the modified Bessel functions of the second kind $K_0(x)$, $K_1(x)$ and the integral $\bar{K}_0(x)=\int_0^x K_0(t)dt$ for $x>1$. These approximations involve (besides exponentials, roots and trigonometric functions) rational approximations to certain auxiliary functions with a maximum error not exceeding 25×10^{-10} in all cases. *U. W. Hochstrasser* (Lawrence, Kan.).

Chambers, Ll. G. Note upon the numerical evaluation of limits of sequences. Math. Tables Aids Comput. **11** (1957), 19–21.

Healy, M. J. R. A rotation method for computing canonical correlations. Math. Tables Aids Comput. **11** (1957), 83–86.

If A and B are symmetric, and

$$\begin{pmatrix} A & C \\ C' & B \end{pmatrix}$$

symmetric and positive definite, the problem is to find matrices U and V with $UAV'=I$, $VBV'=I$, $UCV'=R$ a diagonal matrix. By expressing $A=HH'$, $B=KK'$ and writing $S=UH$, $T=VK$, it turns out easily that S and T are orthogonal and that $SH^{-1}CK'^{-1}T'$ is non-negative diagonal. Then S and T can be generated by an infinite sequence of plane rotations. This type of diagonalization has been discussed by Kogbetliantz for an arbitrary complex matrix [Proc. Internat. Congress Math., 1954, Amsterdam, v. 2, Noordhoff, Groningen, 1954, pp. 356–357]. *A. S. Householder* (Oak Ridge, Tenn.).

Kron, Gabriel. Improved procedure for interconnecting piece-wise solutions. J. Franklin Inst. **262** (1956), 385–392.

This paper concerns the method of "tearing" for the solution of a system of linear equations given by Kron [same J. **259** (1955), 307–333; MR **16**, 864]. It is shown here that some of the steps can be omitted in a purely numerical solution. *R. J. Duffin* (Pittsburgh, Pa.).

Sabroff, Richard R.; and Higgins, T. J. A critical study of Kron's method of "tearing". Matrix Tensor Quart. **7** (1957), 107–113.

It is the purpose of these articles to present a comprehensive, clearly written, correctly detailed development of this method which can be studied by the engineer in practice or teaching who desires to learn the theory of diakoptics (tearing) and use it for the solution of problems in practice. *From the authors' summary.*

Head, J. W. Factorizing polynomials from two rounds of Lin's reduced penultimate remainder process with a real divisor. Aircraft Engrg. **29** (1957), 184–185.

Popoviciu, Tiberiu. Sur une généralisation de la formule d'intégration numérique de Gauss. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **6** (1955), 29–57. (Romanian. Russian and French summaries)

Let $A[f]$ be an additive and homogeneous real-valued functional defined for all real polynomials $f=f(x)$ of the variable x. Let there be given n distinct real points $x_1, \cdots, x_n$ (the knots) and corresponding natural numbers $r_1, \cdots, r_n$ (the multiplicities). Given $f(x)$ let

$$(1)\qquad \varphi(x)=\sum_{i=1}^{n}\sum_{j=0}^{r_i-1}\varphi_{i,j}(x)f^{(j)}(x_i)$$

be the Lagrange-Hermite interpolation polynomial of degree at most $p=r_1+\cdots+r_n-1$ which agrees with $f(x)$ at the knots with the corresponding multiplicities. Many approximation formulae are obtained by using $A[\varphi]$ as an approximation to $A[f]$. From (1), setting $c_{i,j}=A[\varphi_{i,j}]$ we obtain the approximation formula

$$(2)\qquad A[f]\approx\sum_{i=1}^{n}\sum_{j=0}^{r_i-1}c_{i,j}f^{(j)}(x_i).$$

This relation is evidently exact if the degree of f does not exceed p. The relation (2) is called of Gaussian type provided that it is exact for every polynomial of degree $\leqq p+n$. This is the generalization announced in the title. The functional $A[f]$ is said to be of order of positivity k provided that $A[Q^2]>0$ for every real polynomial Q of degree $\leqq k-1$ which does not vanish identically. The main result of this interesting paper is the following theorem. Let $A[f]$ be a functional of the order of positivity k and

let the integer n be given, as well as the n odd integers $r_1, \cdots, r_n$ such that (3) $r_1+\cdots+r_n+n<2k$. Then there exists a corresponding set of distinct knots $x_1, \cdots, x_n$ such that the formula (2) is of Gaussian type. For functionals $A[f]$ which are positive to any given order k, the inequality condition (3) may be disregarded. In this case and when all odd multiplicities r are equal to each other the result was previously obtained by P. Turán [Acta Sci. Math. Szeged **12** (1950), Pars A, 30–37; MR **12**, 164]. Of importance in the proof are the functionals of the special form (4) $A[f]=\sum_{i=1}^{k}\lambda_i f(y_i)$ ($\lambda_i>0$, y_i distinct). The existence of Gauss-type formulae is first shown for the class of functionals (4); later a general $A[f]$ is shown to be identical with an appropriate functional (4) for the class of polynomials of degree $\leqq 2k-1$. The existence proof for a functional (4) is based on the following minimum problem: Given the functional (4), natural integers $n_1, \cdots, n_q$ and positive numbers $s_1, \cdots, s_q$ all >1, to minimize $A[\prod_{i=1}^{q}|\pi_i|^{s_i}]$, where $\pi_i=x^{n_i}+\cdots$ is an arbitrary real polynomial of degree n_i with highest coefficient unity. Several examples of new Gaussian-type formulae conclude the paper. *I. J. Schoenberg.*

Ionescu, D. V. Formules de cubature, le domaine d'intégration étant un triangle quelconque. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Şti. Ser. I. **6** (1955), 7–49. (Romanian. Russian and French summaries)

Crabtree, L. F.; and Woollett, E. R. A new method for the solution of a differential equation with two-point boundary conditions applied to the compressible boundary layer on a yawed infinite wing. J. Roy. Aero. Soc. **60** (1956), 808–809.

A method is presented for the integration of

$$2f_1'f_2'-f_1f_2''-f_2f_1''-f_2'''=1-g_1^2, \tag{1}$$

where f_1 and g_1 are known, tabulated functions and the boundary conditions on f_2 are $f_2(0)=f_2'(0)=f_2'(\infty)=0$. The method is based on expressing f_2 by a Lagrangian interpolation formula and thus reducing equation (1) to a set of linear algebraic equations. This set was solved on the Pilot automatic computing machine at the National Physical Laboratory. The method can be generalized to higher-order equations with two-point boundary conditions.

Equation (1) arose in the calculation of boundary-layer flow of a compressible fluid over a yawed cylinder by L. F. Crabtree [Aero. Quart. **5** (1954), 85–100; MR **16**, 191]. The values of f_2 in the earlier reference are in error. *W. R. Sears* (Ithaca, N.Y.).

Schröder, Johann. Über das Differenzenverfahren bei nichtlinearen Randwertaufgaben. II. Z. Angew. Math. Mech. **36** (1956), 443–455. (English, French and Russian summaries)

[For part I see same Z. **36** (1956), 319–331.] The paper estimates the error in the finite difference solution of the boundary value problem for the second order ordinary differential equation

$$\phi''+f(x, \phi, \phi')=0$$

with linear boundary conditions. The error involves estimates of the higher derivatives of ϕ, which are also obtained in this paper. A principal technique is the use of the "monotonic type" of the system of difference equations satisfied by the error (analogous to a "maximum principle"). *E. Isaacson* (New York, N.Y.).

Salzer, Herbert E. Numerical integration of $y''=\phi(x,y,y')$ using osculatory interpolation. J. Franklin Inst. **263** (1957), 401–409.

In the numerical solution of differential equations of the prediction form $y''=\phi(x, y, y')$, previous methods of and verification effectively use Lagrangian interpolation formulae, calculating the "next" y, for example, from a quadrature formula involving pivotal values of y' only. This paper suggests the advantage of using Hermite's interpolation formula, calculating the next y, for example, from pivotal values of y' and y'', or even of y, y' and y'', a more powerful choice. A variety of formulae and procedures is discussed, and there is a special note on the equation $y''=\phi(x, y)$. The fact that the new formulae are exact for polynomials of higher degree than those used in previous methods indicates the possibility of using a much larger interval size, with consequent saving in the number of steps and total computing labour. *L. Fox.*

Bolton, H. C.; and Scoins, H. I. Eigenvalue problems treated by finite-difference methods. II. Two-dimensional Schrödinger equations. Proc. Cambridge Philos. Soc. **53** (1957), 150–161.

This is the second part of a paper, published in same Proc. **52** (1956), 215–229 [MR **18**, 72]. It deals with the equation $(-\nabla^2+V(x, y))\psi(x, y)=\lambda\psi(x, y)$ with $\psi=0$ on a simple closed boundary. The difference method

$$-u(x-h, y)-u(x+h, y)-u(x, y-h)-\\ u(x, y+h)+[4+h^2V]u(x, y)=\mu u(x, y)$$

is applied with x, y running through the grid points of a grid with mesh size h. It is assumed that the foregoing difference equation admits a solution for continuously varying values of x, y, and that $\mu=h^2\sum_n \nu_n h^n$; $u=\sum_n h^n\phi_n(x, y)$ is valid with $\nu_0=\lambda$, $\phi_0=\psi$. The formulas $\nu_1=0$, $\nu_3=0$,

$$\nu_2\iint\phi_0^2dxdy=\\ -\frac{1}{12}\iint(\nabla^2\phi_0)^2dxdy+\frac{1}{6}\iint\phi_0\Big(\frac{\partial^4}{\partial x^2\partial y^2}\phi_0\Big)dxdy$$

are derived, and it is concluded that the eigenvalues Λ of the difference method follow the law $\Lambda(h)=\lambda+\nu_2h^2+O(h^4)$. The reviewer fails to see that this reasoning will apply to the case of general non-rectangular boundaries. However, the examples which are presented refer to square domains with boundaries $x=0, 1$; $y=0, 1$. The examples deal with the Schrödinger equation of two electrons in a sphere and with the helium atom. The numerical calculations use an extrapolation to $h^2=0$ for the eigenvalues by means of the Neville-method. The difference equations are solved by relaxation; the relative merits of different relaxation processes are discussed. *H. Bückner.*

Rose, Milton E. On the integration of non-linear parabolic equations by implicit difference methods. Quart. Appl. Math. **14** (1956), 237–248.

The equation considered is $\partial^2u/\partial x^2=F(x, t, u, u_x, u_t)$, in the region $0\leqq x\leqq L$, $0\leqq t$. The value of u is prescribed at $t=0$ and on the lateral boundaries linear relations between u and u_x are specified. The difference scheme considered replaces the second space derivative by centered differences, averaged with weights w_1 and w_2 at times $n+1$ and n, and u_t is replaced by a forward time difference. Under the assumption that $w_2\lambda$ is less than $a/2$, where a denotes a lower bound for F_{u_t} and λ abbreviates

$\Delta t/(\Delta x)^2$, the maximum principle holds. Based on this, the author proves convergence as Δt, Δx tend to zero. Next the nonlinear difference equation is solved by iterating the transformation

$$v=u+\delta L,$$

where u is the old value, v the new value and L is the difference operator. In evaluating L the latest information is used; δ has to be chosen smaller than a quantity determined by λ and w_1. *P. D. Lax* (New York, N.Y.).

Crank, J. Two methods for the numerical solution of moving-boundary problems in diffusion and heat flow. Quart. J. Mech. Appl. Math. **10** (1957), 220–231.

"This paper describes two new ways of dealing with the general problem of a moving boundary. In the first method the boundary is rendered stationary by an appropriate change of variable. This means that instead of solving the usual diffusion or heat-conduction equation with a moving boundary we have essentially an eigenvalue problem with fixed boundaries. The equation to be solved contains parameters, associated with the movement of the boundary, for which values have to be determined such that the boundary conditions are satisfied. In the second method Lagrangian interpolation formulae for non-equal intervals are introduced into finite-difference formulae in order to follow the movement of the boundary."

The paper presents the two methods quite clearly and the first method is illustrated by a specific set of numerical results. *R. W. Hamming* (Murray Hill, N.J.).

Greenspan, Donald. On a "best" 9-point difference equation analogue of Laplace's equation. J. Franklin Inst. **263** (1957), 425–430.

The author considers 9-point finite difference equation analogues of Laplace's equation of the form $\sum_{i=0}^{8}\alpha_i u_i=0$, where $u_i=u(x_i, y_i)$ and $|x_i-x_0|+|y_i-y_0|=h$ $(i=1, 2, 3, 4)$, $|x_i-x_0|+|y_i-y_0|=2h$ $(i=5, 6, 7, 8)$, and where h is the mesh size. It is shown that the usual 9-point formula, $-20u_0+4\sum_{i=1}^{4}u_i+\sum_{i=5}^{8}u_i=0$, is of seventh order and that there does not exist an eight order approximation such that $\lim_{h\to 0}\alpha_i\neq 0$ for at least one i.

The following corrections should be noted: several times the summation is taken from 0 to 9 instead of from 0 to 8; according to the usual definitions of the symbols o and O, the former should be replaced by the latter in several places. *D. M. Young, Jr.* (Los Angeles, Calif.).

Sofronov, I. D. On approximate solution of singular integral equations. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 37–39. (Russian)

The author discusses two methods of numerical solution of singular integral equations of the form

$$\text{(1)}\quad a(x)\varphi(x)+\frac{1}{2\pi}\int_0^{2\pi}n(t, x)\cot\tfrac{1}{2}(t-x)\varphi(t)\,dt=f(x),$$

where $a(x)$, $n(t, x)$ and $f(x)$ satisfy Hölder conditions of order α.

In the first method (1) is replaced by the system of equations

$$a_i\varphi_i+\sum_{k=0}^{n-1}a_{ik}\varphi_k=f_i\ (0\leq i\leq n-1),$$

where $x_i=ih$, $h=2\pi n^{-1}$, $a_i=a(x_i)$, $f_i=f(x_i)$,

$$a_{ik}=\frac{1}{2\pi}\int_{x_{k-1}}^{x_{k+1}}n(t, x_i)\cot\tfrac{1}{2}(t-x_i)\eta_k(t)\,dt,$$

$$\eta_k(t)=h^{-1}(t-x_{k-1})\ (t\leq x_k),\ \eta_k(t)=h^{-1}(x_{k+1}-t)\ (t\geq x_k),$$

the given functions being extended to have period 2π. The numbers a_{ii} are defined as principal value integrals. The approximate solution is then the piecewise linear function taking the value φ_i at x_i.

In the second method (1) is replaced by

$$a_i\varphi_i+\frac{2}{2n+1}\sum_{k=0}^{2n}{}'n_{ik}\cot\tfrac{1}{2}(x_k-x_i)\varphi_k=f_i\ (0\leq i\leq 2n-1),$$

where $x_i=ih$, $h=\pi n^{-1}$, $n_{ik}=n(x_i, x_k)$, $a_i=a(x_i)$, $f_i=f(x_i)$, and Σ' denotes summation over those values of k for which $i-k$ is odd. The approximate solution is then the trigonometric interpolation polynomial taking the value φ_i at x_i.

In both cases it is shown that if (1) has a unique solution for every f, then the approximating system of equations has a unique solution for all sufficiently large n, and an estimate is given for the difference between the approximate and exact solutions. The theory given in the author's earlier paper [same Dokl. (N.S.) **110** (1956), 940–942; MR **18**, 906] is used. Reasons are given for preferring the second method, especially in cases when $\varphi(x)$ can be expected to be several times differentiable. There are no proofs.

F. Smithies (Cambridge, England).

Berg, Lothar. Lösungsverfahren für singuläre Integralgleichungen. I. Math. Nachr. **14** (1955), 193–212 (1956).

With reference to the definitions, notations and results of the book by N. S. Muskhelishvili, "Singular integral equations" [OGIZ, Moscow-Leningrad, 1946; MR **8**, 586; **15**, 434] the author proposes to develop a numerical method for solving directly a singular integral equation $K\phi=f$ with f being orthogonal to the null-elements of the adjoint singular operator K'. The equation is defined for the points s, t of the boundary L of a finite and connected domain of the complex plane. L is assumed to consist of a finite number of smooth contours.

In order to find solutions numerically the iterative process $\phi_{n+1}(s)=\phi_n(s)+\alpha_n(s)\cdot I\phi_n$; $I\phi_n=f-K\phi_n$ is studied. The coefficients α_n can be selected in special ways. The author refers to some of the choices which have already been used for Fredholm integral equations, especially to constant coefficients and to the case $\alpha_n\equiv 1$, which corresponds to the classical Neumann-Liouville iteration. He derives sufficient conditions of convergence; he also deals with the method of steepest descent, in the sense that the integral $\int_L|I\phi_{n+1}|^2ds$ is minimized by a suitable choice of α_n. Nonconstant coefficients α_n are set up as linear combinations of a given set of functions. The case of a singular equation with a nonlinear term is briefly discussed at the end of the paper.

The paper also contains a theoretical result about operators M which regularize K, such that $MK\phi=Mf=g$ leads to an equation with a regular kernel in the sense of Muskhelishvili. It is proved that M can be selected in such a way that all solutions of the regularized equation give all solutions of the original equation. The resolvent of the regularized equation is discussed and an analogue with the functional equations for ordinary Fredholm resolvents established. *H. Bückner.*

Mysovskih, I. P. Estimation of error arising in the solution of an integral equation by the method of mechanical quadratures. Vestnik Leningrad. Univ. **11** (1956), no. 19, 66–72. (Russian)

The following process for finding an approximate so-

lution of an integral equation of the form

$$\varphi(s)=\lambda\int_a^b K(s,t)\varphi(t)dt+f(s) \tag{1}$$

is discussed. The integral in (1) is approximated by a quadrature formula

$$\sum_{k=1}^n A_k K(s,t_k)\varphi(t_k),$$

equation (1) thus being replaced by

$$\bar\varphi(s)=\lambda\sum_{k=1}^n A_k K(s,t_k)\bar\varphi(t_k)+f(s). \tag{2}$$

The system of equations obtained by putting $s=t_i$ ($1\leq i\leq n$) in (2) is solved, and (2) then gives an approximation $\bar\varphi(s)$ to the solution of (1). The author obtains an estimate of the error $\max|\varphi(s)-\bar\varphi(s)|$ of this solution in terms of the quadrature errors

$$\varepsilon_f(s)=\int_a^b K(s,t)f(t)dt-\sum_{k=1}^n A_k(s,t_k)f(t_k),$$

$$\varepsilon(s,t)=\int_a^b K(s,t')K(t',t)dt'-\sum_{k=1}^n A_k K(s,t_k)K(t_k,t).$$

Two numerical examples are given to show that this estimate is reasonably close to the truth, and is better than that given by certain other methods. The author remarks that his method is applicable even when the kernel has discontinuous derivatives, a case that often arises in practice. *F. Smithies* (Cambridge, England).

★ **Kuntzmann, J. Evaluations d'erreur dans les représentations approchées de derivées.** Société d'Electronique et d'Automatisme, Courbevoie (Seine), France, 1955. 49 pp. (mimeographed)

This supplements an earlier report by the same author [Formules de dérivation approchée au moyen de points équidistants, Société d'Electronique et d'Automatisme, Courbevoie, Rep. Tech. 1 373/1 (1954); MR **17**, 1009] which gave, in Lagrangian form, expressions approximating derivatives. The principal part of the error at each point is given, and an indication whether the formula is definite (in the sense of W. E. Milne [Numerical calculus, Princeton, 1949; MR **10**, 483] and the author [C. R. Acad. Sci. Paris **239** (1954), 1110–1111; MR **16**, 404]) or not. In the indefinite case, i.e., when the kernel in the integral representation of the error does change sign, the integral of the absolute value of the kernel has been calculated. In both cases, therefore, the error is easy to estimate.

The tables cover the cases of n-point interpolation, $2\leq n\leq 11$, and deal with the errors at each point $x_i=x_0+ih$ ($0\leq i\leq n-1$) for each derivative D^r ($1\leq r\leq n-1$). The numerical coefficients are given as rational numbers when possible and otherwise to 3D, using a calculation of the zeros of the kernel to within .01. *John Todd.*

Kruskal, William. On the note "On the propagation of error by multiplication" by Perry and Morelock. Amer. Math. Monthly **64** (1957), 254–255.

Wheeler, Roger F. Solving quadratics quickly. Math. Gaz. **41** (1957), 98–101.

The author presents a method of constructing nomograms for solving the quadratic equations $ax^2+x+b=0$. It consists of two parallel straight lines graduated linearly as scales for a and b, and a circle, tangent to boht lines, on which the real solutions of the equations can be read. *S. Kulik* (Columbia, S.C.).

Askovitz, S. I. A short-cut graphic method for fitting the best straight line to a series of points according to the criterion of least squares. J. Amer. Statist. Assoc. **52** (1957), 13–17.

A simple technique is presented for obtaining without calculation the best fitting line according to the least squares criterion, provided the points are equally spaced horizontally. A graphic measure of residual variability is also derived. *Author's summary.*

Pell, W. H. Graphical solution of single-degree-of-freedom vibration problem with arbitrary damping and restoring forces. J. Appl. Mech. **24** (1957), 311–312.

L'équation $x''+\varphi(x')+f(x)=0$ s'intègre graphiquement d'une manière très simple dans le plan de phase. *J. Kuntzmann* (Grenoble).

Nyström, E. J. On ellipsographs. Nordisk Mat. Tidskr. **5** (1957), 19–28, 64. (Norwegian. English summary)

Eight kinds of mechanisms for drawing ellipses.

Popov, A. A. Numerical harmonic analysis using focal ordinates. Inžen. Sb. **23** (1956), 214–230. (Russian)

A method of numerical harmonic analysis of periodic functions given approximately by broken straight lines is presented in this paper. Use is made of special points called orthogonal focuses by the author [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **38** (1943), 286–288; Method of orthogonal foci in structural mechanics, Gosstrolizdat, Moscow, 1953; MR **5**, 160]. *S. Kulik* (Columbia, S.C.).

★ **Карпов, К. А.; и Разумовский, С. Н. [Karpov, K. A.; and Razumovskiĭ, S. N.] Таблицы интегрального логарифма. [Tables of the integral logarithm.]** Izdat. Akad. Nauk SSSR, Moscow, 1956. 319 pp. 33.30 rubles.

This is a table of li $x=\int_0^x dt/\ln t$ to 7S for

$$x=0(.0001)2.5(.001)20(.01)200(.1)500(1)1000(10)25000.$$

It is stated that the results are correct to within .6 of a unit in the last place. The table was computed on the БЭСМ [see S. A. Lebedev, J. Assoc. Comput. Mach. **3** (1956), 129–133; MR **18**, 339], using Simpson's rule.

In general, linear interpolation gives an error not exceeding 1.6 units. However, in the ranges (0, .0403), (.95, 1.05) and (1.4477, 1.4551), the latter including the zero $1.4513\cdots$ of li x, other methods are required. For the intervals (.0017, .0403) and (1.4477, 1.4551), quadratic interpolation gives an error not exceeding 1.8 units. To facilitate this, first and second differences are given in the range (0, .05) and a table of $\frac{1}{2}t(1-t)$ to 4D for $t=0(.01).5$ is provided. In the interval (.95, 1.05), linear interpolation in the auxiliary table of li $x-\ln|1-x|$, given to 7D for $x=.95(.0001)1.65$, is satisfactory. In the ranges (.00000031, .0017) and (25000, 3269000), it is suggested that values be obtained from tables of the exponential integral in the same series [Tables of the function $w(z)=e^{-z^2}\int_0^z e^{x^2}dx$ in a complex region, Izdat. Akad. Nauk SSSR, Moscow, 1954; MR **16**, 749], using li $x=\mathrm{Ei}(\ln x)$. For (0, .00000031) and (3269000, ∞), use of the asymptotic series is recommended. There are worked examples showing the methods in use in different ranges.

The main table gives 500 values on an opening, no differences being given; in the range (0, .5), there are two columns on each page, each giving 50 values of li x and the first two differences. The table is clearly printed. *John Todd* (Pasadena, Calif.).

★ **Clark, George C.; and Churchill, Stuart W. Tables of Legendre polynomials.** Engineering Research Institute, University of Michigan, 1957. ix+92 pp. $4.50.

This table gives the Legendre polynomials $P_n(\cos\theta)$, $n=1(1)80$, for $\theta=1(1)180°$, to 6D. The table was computed on the MIDAC, using the recurrence relation

$$(n+1)P_{n+1}=(2n+1)xP_n-nP_{n-1},$$

for use in connection with engineering applications of light-scattering, briefly described in the introductory material, which also contains a list of references to earlier tables. The booklet has been produced by photo-offset processes from masters prepared directly on the machine. The reproduction is clear.

As a check, the values of $P_{80}(\cos\theta)$ for $\theta=1(1)180°$ were computed on the National Bureau of Standards IBM 704 by E. Brauer and J. C. Cager, using the representation $P_{80}(\cos\theta)=\sum A_n\cos(80-2n)\theta$. There was complete agreement except for a rounding discrepancy at $\theta=14°$. Since this was influenced by the ninth decimal, and the checking programs were only good to between seven and eight, no determination of the correct value has been made. *John Todd* (Pasadena, Calif.).

MacDonald, D. K. C.; and Towle, Lois T. Integrals of interest in metallic conductivity. Canad. J. Phys. **34** (1956), 418–419.

This note gives a table, to 4–6 S, of

$$J_r(x)=\int_0^x z^r(e^z-1)^{-1}(1-e^{-z})^{-1}dz$$

for $r=2, 3, 4, 6$, $x=0.1, 0.25, 0.5, 1.0, 1.2, 1.5, 2(1)6, 8, 10, 13, 20$ and ∞. A number of non-trivial errors have been discovered by W. F. Cahill [Math. Tables Aids Comput. **11** (1957), 38–39]. *John Todd* (Washington, D.C.).

★ **Fröberg, Carl-Erik. Complete elliptic integrals; Lund University, Department of Numerical Analysis, Table No. 2.** CWK Gleerup, Lund, 1957. 82 pp. 10 kr.

The integrals K, K', E, E' and the auxiliary functions $S=K'-\log 4k^{-1}$, $S'=K-\log 4(k')^{-1}$ are tabulated to 10D for $k=0(.001).9(.0001)1$. Second central differences, modified when necessary, are given, except where they would be insufficient: near zero for K', E', S and near unity for K, E, S'. The values of K, K', E, E' to 10D for $k=3-2\sqrt{2}$, $\sin 15°$, $\sqrt{2}-1$, $\sin 45°$ are also given.

The basic functions K, E were computed in two ways. The first was by repeated use of the Landen transformation, reducing the $k=k_0$ to such a value k_n for which it may assumed that $K_n=E_n=\frac{1}{2}\pi$, and then working back to K and E. The second method was by power series $P(k^2)$ for small k or by an expansion of the form

$$K=\log 4(k')^{-1}P_1(k'^2)+P_2(k'^2)$$

for k near unity. The whole table was prepared on SMIL, the electronic computer of Lund University. All values were checked using the Legendre identity, and it is stated that errors in the tabulated values will not exceed a unit in the last place.

Instructions for interpolation are given: Everett's formula is recommended where the second differences are tabulated, and special methods for the use at the ends of the range are given.

The tables are clearly printed. *John Todd.*

See also: Cheema, p. 17; Frame, p. 22; Moonan, p. 70; Sakoda and Cohen, p. 73; Seames and Conway, p. 80; Bader, p. 89; Vernotte, p. 94; Tárczy-Hornoch, p. 105; Markowitz and Manne, p. 106.

Computing Machines

Linsman, M. Le choix du code dans la construction des machines mathématiques décimales. Bull. Soc. Roy. Sci. Liège **25** (1956), 608–635.

After a brief discussion on the factors entering into the choice of a code in the design of computing machines, and of the relative merits of pure binary and coded decimal systems, coded systems requiring four bits per digit are examined. Tables are given showing the number of codes with weighted digits, firstly where the weights are all positive, and then where negative weights are permitted. Next some codes which are constructed by selecting ten of the binary representations for the numbers 0, 1, $\cdots$, 16, are examined. A definition of elements which are connected (lié) is given and it is shown how this serves to restrict the number of possibilities. Codes which are biquinary in the sense that they make use of the fact that $10=2\times 5$ are examined, and finally codes are enumerated where the number of 1's which can arise in the representation of the digits 0, $\cdots$, 9 is minimal. *C. C. Gotlieb* (Toronto, Ont.).

Newman, E. A.; and Wright, M. A. An automatic floating-address machine. Proc. Inst. Elec. Engrs. B. **103** (1956), supplement no. 1, 134–137.

In order to speed use of a hierarchy of storage in a general purpose digital computer, the authors have included in the design of the T.A.C.T. (Three-Address Code machine, Tagged) a procedure for storing names of data along with the data. These "tags" are not "floating addresses" in the classical computer sense in that they are names for data only, and not instructions. The machine has sixteen 48-bit words of high-speed storage of a type not specified, and 1024 words of magnetic drum storage. Each word consists of a 36-bit data portion and a 12-bit tag. On processing an instruction, the machine searches the high-speed memory for the needed tags, and upon finding them sends the corresponding data to the arithmetic unit. Results are stored similarly in that location containing a specific tag. When a tagged variable is not in the high-speed memory, an arbitrary word in that memory is replaced by the correct word from the magnetic drum memory. Intuitively this process would appear to speed up performance of operations when instructions are stored sequentially on the drum. Logical diagrams are included for the circuitry involved in manipulation of the tags. *J. W. Carr, III* (Ann Arbor, Mich.).

Worthy, W. D. Use of interpretation routines on a general-purpose digital computer for the design of linear and non-linear control systems. Proc. Inst. Elec. Engrs. B. **103** (1956), supplement no. 1, 68–76.

A convenient yet versatile digital computer program is described which facilitates the design of control systems whose behavior may be described by a set of simultaneous ordinary differential equations. This program is most easily applied to the usual case in which the first derivative of each variable may be expressed as the sum of relatively few linear terms and a constant. The equations are integrated numerically by the computer using a

second-order version of the Runge-Kutta integration process which requires two evaluations of the derivatives per step. Machine instructions may be included among the coefficients of terms in the equations so as to permit the formation of nonlinear terms if they are required to describe the system. Various options allow the user to make changes in one or more of the coefficients and repeat the integration process. Graphical or printed output may be obtained for any variable.

This program is compared with mechanical and electrical differential analyzers as regards speed, accuracy, convenience and versatility.

D. E. Muller.

Wagner, Harvey M. A lemma for automatic optimum programming on the IBM 650. Math. Tables Aids Comput. 11(1957), 101–104.

The lemma is as follows: Let n and k be positive integers. Then the sequence a) $1, 1+n, 1+2n, 1+3n, \cdots$ modulo $(nk+1)$ completely exhausts the numbers $(1, 2, 3, \cdots, nk+1)$ before repeating any number in the sequence; similarly for b) $1, 1+n, 1+2n, 1+3n, \cdots$ modulo $(n(2k+1)+2)/2$, where n is even; and c) $1, 1+n, 1+2n, 1+3n, \cdots$ modulo $(n(2k+1)+1)/2$, where n is odd.

In writing an optimized programmed for a computer, the problem is to locate the instructions in the store (usually a drum) so that they become available to the read-out device just as the machine needs them. It is easy to do this for the instructions in the first part of a program, but as the store is filled, it becomes increasingly difficult to do it later. This paper describes a technique, suggested by the lemma, for placing all the instructions optimally. The actual rearrangement of a program with instructions and data in consecutive locations to one where they are arranged optimally is done by the computer. *C. C. Gotlieb* (Toronto, Ont.).

Birtwistle, B.; and Dent, Beryl M. The digital computer as an aid to the electrical design engineer. Proc. Inst. Elec. Engrs. B. 103 (1956), supplement no. 1, 47–53.

Oldfield, J. V.; McDonald, D.; and Davies, M. W. Humphrey. Transformer design with digital computers. Proc. Inst. Elec. Engrs. B. 103 (1956), supplement no. 1, 54–58.

Peterson, W. W. Addressing for random-access storage. IBM J. Res. Develop. 1 (1957), 130–146.

Thomson, W. T. Analog computer for nonlinear system with hysteresis. J. Appl. Mech. 24 (1957), 245–247.

This paper describes an analogue computer for solving piece-wise linear differential equations describing the behaviour of structural materials loaded beyond the elastic limit. For extensions within the elastic limit the normal (linear) differential equation for damped harmonic motion applies; for (positive) extension beyond this limit a lower restoring force represents the regime of plasticity; motion during any reduction in extension from the current maximum is subject to the original elastic restoring forces. In the computer these variations in restoring forces are achieved by two relays switched by a displacement exceeding the current elastic limit and by the sign of the velocity of displacement respectively. A specimen response is given showing the result of introducing the hysteresis effect of elastic return.

J. G. L. Michel (Teddington).

Lafon, R. Calculateur analogique destiné au dépouillement des mesures de température des flammes de fusées. Rech. Aéro. no. 57 (1957), 23–29.

Cet article décrit un calculateur destiné à être associé à un équipement de mesures. L'opération, essentiellement un quotient, est effectuée grâce à des générateurs de logarithmes constitués par des réseaux à diodes.

Résumé de l'auteur.

Flodmark, Stig. Note on a standard program for calculation of one-electron molecular integrals of overlap type by use of the Swedish electronic computor BESK. Ark. Fys. 11 (1957), 417–419.

See also: Robinson, p. 14; Crabtree and Woollett, p. 65; Thurston, p. 81.

PROBABILITY

★**Onicescu, O.; Mihoc, G.; and Ionescu Tulcea, C. T. Calculul probabilităţilor şi aplicaţii. [The calculus of probability and its applications.]** Editura Academiei Republicii Populare Romine, 1956. 787 pp. Lei 26,30.

There are three parts. The first part is classical. The second, entitled "Stochastic processes", deals with Markov chains, ergodic problems, distributions, limit theorems, time series, aleatory mechanics, and mixing processes. The third part, entitled "Applications", includes statistical mechanics, mathematical statistics, demography and actuarial theory (13 pages are devoted to insurance other than life insurance), and finally there are 6 pages on stellar statistics. There is a bibliography of 221 titles.

Kolmogorov, A. N. On the Skorohod convergence. Teor. Veroyatnost. i Primenen. 1 (1956), 239–247. (Russian. English summary)

Consider the space D of functions f from $T=[0, 1]$ to a metric space X having only discontinuities of the first kind, namely such that at each t, $f(t-0)$ and $f(t+0)$ exist. By intervention of the graph space $T\times X$ it is possible to convert D into a complete metric space. Not all details are given, and there is a misprint on p. 244 regarding the definition of ω^*. *K. L. Chung.*

Rényi, A. On conditional probability spaces generated by a dimensionally ordered set of measures. Teor. Veroyatnost. i Primenen. 1 (1956), 61–71. (Russian summary)

The main result is a weaker (but more easily proved) version of a theorem of A. Császár [Acta Math. Acad. Sci. Hungar. 6 (1955), 337–361; MR 18, 340]. The paper is semi-expository. *D. A. Darling* (Ann Arbor, Mich.).

Dall'Aglio, Giorgio. Sulla regressione pseudo-lineare. Rend. Mat. e Appl. (5) 15 (1956) 453–468 (1957).

Let X and Y be two random variables and suppose that the conditional expectation $E(Y|X)$ of Y given X, as well as all the moments of X, exist. A polynomial $p_r(x)$ of degree r is called a least squares polynomial of degree r if it minimizes the expectation $E\{[Y-p_r(X)]^2\}$. The random variable Y is said to have pseudo-linear regression on X if the regression polynomials of any degree r coincide with one and the same straight line. The linearity of the

regression of Y on X implies pseudo-linearity. The author constructs an example to show that the converse is in general not true and derives conditions for the equivalence of the concepts of linear regression and of pseudo-linear regression. *E. Lukacs* (Washington, D.C.).

Walsh, John E. Validity of approximate normality values for $\mu \pm k\sigma$ areas of practical type continuous populations. Ann. Inst. Statist. Math., Tokyo **8** (1956), 79–86.

The author makes the empirical observation that the fraction of a continuous distribution contained in the interval $(\mu-k\sigma, \mu+k\sigma)$, where μ and σ are the mean and standard deviation, often appears to be nearly equal to the value obtained under the assumption of normality. He investigates this remark for the class of distributions with frequency functions which can be "adequately" represented by the first seven terms of their Edgeworth series expansions. He finds that for certain values of the parameter k, notably $k=\sqrt{3}$, the empirical relation is roughly valid for these distributions. *J. H. Curtiss.*

Good, I. J. The surprise index for the multivariate normal distribution. Ann. Math. Statist. **27** (1956), 1130–1135.

An event is surprising if its probability is small in comparison with the expected value of the probability; the surprise index is $\lambda_1=E(p_i)/p_i$ [see W. Weaver, Sci. Monthly **67** (1948), 390–392]. For continuous distributions the author introduces the indices

$$\lambda_u=[E(p^{*u})]^{1/u}/p,\ \lambda_0=\exp[E(\log p^*)-\log p]$$

and extends to general distributions by use of the multiplicative properties. The value and distribution of $\Lambda_u=\log \lambda_u$ are found explicitly for the multivariate normal distribution, and the results lead the author to believe that λ_0 or Λ_0 is a more natural measure of surprise than λ_1 or Λ_1. Comparison with other statistical procedures is given. {The first equation on p. 1132 defines λ_u, not λ_0. The author writes: "The remark about Hotelling's generalized student test at the end of the paper is misleading and should be deleted."} *R. M. Redheffer.*

Moonan, William J. Linear transformation to a set of stochastically dependent normal variables. J. Amer. Statist. Assoc. **52** (1957), 247–252.

The author presents a method for transforming a set of independent normally distributed variables with unit variances into a set jointly normally distributed with prescribed covariance matrix. Mathematically the problem consists of finding a triangular matrix A such that $M=AA'$, where M is a prescribed positive definite matrix. *K. J. Arrow* (Stanford, Calif.).

Petrov, V. V. A local theorem for densities of sums of independent random variables. Teor. Veroyatnost. i Primenen. **1** (1956), 349–357. (Russian. English summary)

Castoldi, Luigi. "Riducibilità" di ogni distribuzione statistica multipla. Rend. Sem. Fac. Sci. Univ. Cagliari **25** (1955), 137–142 (1956).

The author states that, for any joint probability distribution of $(X_1, X_2, \cdots, X_n)$, there exist infinitely many transformations $X_i=X_i(\xi_1, \xi_2, \cdots, \xi_n)$ with non-vanishing Jacobian such that $\xi_1, \xi_2, \cdots, \xi_n$ are totally independent, and he indicates a procedure for obtaining all such transfromations. *Z. W. Birnbaum* (Seattle, Wash.).

Heppes, A. On the determination of probability distributions of more dimensions by their projections. Acta Math. Acad. Sci. Hungar. **7** (1956), 403–410. (Russian summary)

A discrete n-dimensional distribution consisting of k arbitrary points is uniquely determined by its projections on $k+1$ non-parallel $(n-1)$-dimensional subspaces [cf. A. Rényi, same Acta **3** (1952), 131–142; MR **14**, 771]. For the projections to determine a non-discrete distribution, their number must, as shown for $n=2$, be infinite if the density function satisfies a weak regularity condition. *H. Wold* (Uppsala).

Barton, D. E.; and David, F. N. Multiple runs. Biometrika **44** (1957), 168–178.

Suppose we arrange r_1 balls of a first color, r_2 balls of a second color, $\cdots$, r_k balls of a kth color, in random order on a line. Denote $r_1+r_2+\cdots+r_k$ by r, and let T denote the total number of runs of the k colors. The probability distribution, moments, and cumulants of T are found. Define S as $r-T$. If k is fixed and r increases, it is shown that the distribution of S approaches a normal distribution. If an upper bound is put on r_i, and k and r both approach infinity, it is shown that the distribution of S approaches a Poisson distribution. These results generalize known results for $k=2$.

It is shown that the distribution of S can, for practical purposes, be approximated by a binomial distribution. Applications to tests of hypotheses are discussed. *L. Weiss* (Ithaca, N.Y.).

Watanabe, Hisao. On the Poisson distribution. J. Math. Soc. Japan **8** (1956), 127–134.

The author investigates the limiting distributions of the number of particles lying in an interval of a system of particles whose movements are independent of each other. Poisson distribution is derived for the case of lattice distribution. Non-lattice case was treated before by G. Maruyama [Nat. Sci. Rep. Ochanomizu Univ. **6** (1955), 7–24; MR **18**, 341]. *S. C. Moy.*

Ossoskow, G. A. Ein Grenztheorem für Folgen gleichartiger Ereignisse. Teor. Veroyatnost. i Primenen. **1** (1956), 274–282. (Russian. German summary)

Kac, M. Distribution of eigenvalues of certain integral operators. Michigan Math. J. **3** (1955–1956), 141–148.

The asymptotic distribution of the eigenvalues for the integral equation

$$\int_\Omega \phi(\mathbf{p})|\mathbf{r}-\mathbf{p}|^{-\alpha/2}\,d\mathbf{p}=\lambda\phi(\mathbf{r})\ (0<\alpha\leq 2),$$

where Ω is a 3-dimensional region, is discussed, based upon the theory of Brownian motion, accompanied by a Tauberian theorem and Mercer's theorem. Though the case $\alpha=2$ yielded [cf. Kac, Proc. 2nd Berkeley Symposium on Math. Statist. and Probability, 1950, Univ. of California Press, 1951, pp. 189–215; MR **13**, 568] the classical theorem of H. Weyl concerning the eigenvalues of the Laplacian, the general case $0<\alpha<2$ has no equivalent formulation in terms of a differential equation; it has rather a bearing on the stable processes. The method of proof is an adaptation of the cited paper and is illustrated in the one-dimensional case $\int_{-a}^{a}\phi(y)|y-x|^{-\alpha}V(y)dy=\lambda\phi(x)$, where $V(y)>m>0$ on $[-a, a]$, and the corresponding result reads

$$\lambda_n^{-1}\sim n^{1-\alpha}\left(\pi^{-1}\int_{-a}^{a} V(x)^{1/(1-\alpha)}dx\right)^{\alpha-1}.$$

K. Yosida (Tokyo).

Rozenblat-Rot, M. Entropy of stochastic processes. Dokl. Akad. Nauk SSSR (N.S.) 112 (1957), 16–19. (Russian)

Consider a stochastic process $A=\{x_t\}$ for $t\in I$, I the set of all real numbers. Let $x_t, \cdots, x_{t+n-1}$ have a density $\pi^{[t,t+n-1]}(x^{[t,t+n-1]})$. Let

$$f^{[t,t+n-1]}(x)=-n^{-1}\log \pi^{[t,t+n-1]}(x^{[t,t+n-1]}).$$

The entropy of the process at time t is defined as $H_t(A)=\lim_{n\to\infty} Mf^{[t,t+n-1]}(x)$ when this limit exists. The author states a number of theorems relating to the existence of this limit and to conditions under which $f^{[t,t+n-1]}(x)$ will converge in probability to $H_t(A)$. Conditions under which $H_t(A)$ is independent of t are also given. These results generalize results given by Shannon [Bell System Tech. J. 27 (1948), 379–423, 623–656; MR 10, 133] and McMillan [Ann. Math. Statist. 24 (1953), 196–219; MR 14, 1101]. *J. L. Snell* (Hanover, N.H.).

Shinbrot, Marvin. On a method for optimization of time-varying linear systems with nonstationary inputs. NACA Tech. Note no. 3791 (1956), 39 pp.

Using the procedure of Wiener [Extrapolation, interpolation, and smoothing of stationary time series, Wiley, New York, 1949; MR 11, 118] as modified by Booton [Mass. Inst. Tech. Dynamic Analysis and Control Lab., Meteor. Rep. no. 72 (1951)] for non-stationary stochastic processes, the author seeks an operator to apply to an input which is the sum of a message plus a noise, in order to get a desired output. The familiar least square minimization is used, but averages are with respect to the statistical parameters. The result is the integral equation

$$\phi_{\mu i}(t,\tau)=\int_0^t g(t,\sigma)\phi_{ii}(\tau,\sigma)\,d\sigma,$$

where ϕ_{ii} the autocorrelation of the input and $\phi_{\mu i}$ the cross correlation of the input and desired output. The $g(t,\tau)$ is the desired operator. The author assumes that

$$\phi_{ii}(t,\tau)=\begin{cases}\sum_1^P a_p(t)b_p(\tau) & (t\le\tau)\\ \sum_1^P a_p(\tau)b_p(t) & (t\ge\tau)\end{cases}$$

and

$$\sum_1^P (a_p(t)b_p(\tau)-a_p(\tau)b_p(t))$$

is a function of $(t-\tau)$ only. Also, $\phi_{\mu i}$ is assumed to be $\sum_1^q c_p(t)b_p(\tau)$. He shows under these conditions the equation can be solved for g. A number of examples are considered. *N. Levinson* (Cambridge, Mass.).

Burkholder, D. L. On a class of stochastic approximation processes. Ann. Math. Statist. 27 (1956), 1044–1059.

For each real x let $G(\cdot|x)$ be a distribution function with mean $M(x)=\int y dG(y|x)$. This paper deals with iterative processes $x_1, x_2, \cdots$ for approximating a point θ at which M will have some preassigned property: so-called "stochastic approximation processes." The author first constructs a class A_0 of stochastic processes which contains as subclasses (1) Robbins-Monroe processes, useful when θ is (in an appropriate sense) a point of strict increase of M, (2) Kiefer-Wolfowitz processes, useful when θ is a maximum of M, and (3) a class of processes which can in certain cases be used to locate an inflection point of M. Sufficient conditions are given for convergence with probability one of an A_0 process; this permits a relaxation of the hypotheses in known results on (2), and gives information for (3). Asymptotic normality of] $n^{\zeta}(x_n-\theta)$ is shown, given enough conditions, and further assumptions enable one to redefine the last sequence so that the limiting normal distribution will have parameters (0, 1). Again, this strengthens previous theorems about (1) and (2), and has consequences for (3). The precise statements are rather complicated, and therefore are not reproduced here. *J. Feldman.*

★**Дуб, Дж. Л. [Dub, Dž. L. (Doob, J. L.)] Вероятностные процессы [Stochastic processes.]** Izdat. Inostr. Lit., Moscow, 1956. 605 pp. 35.90 rubles.

A translation by R. L. Dobrušin and A. M. Yaglom from the English of the book reviewed in MR 15, 445.

Statulyavičus, V. A. On a local limit theorem for inhomogeneous Markoff chains. Dokl. Akad. Nauk SSSR (N.S.) 107 (1956), 516–519. (Russian)

The author considers a sequence of s-dimensional random variables $\xi_0, \xi_1, \cdots$ connected in a Markov chain and assuming as values the unit vectors $e_1=(1, 0, \cdots, 0)$, $e_2=(0, 1, \cdots, 0), \cdots, e_s=(0, 0, \cdots, 1)$ of an s-dimensional vector space with transition probabilities

$$p_{ij}^{(k)}=P(\xi_k=e_j|\xi_{k-1}=e_i).$$

The asymptotic behavior of the probabilities

$$P_\gamma(m_1, m_2, \cdots, m_s)=P\{S_n=(m_1, m_2, \cdots, m_s)|\xi_0=e_\gamma\},$$

where $S_n=\sum_{t=1}^n \xi_t$, are studied under the following conditions: (A) $p_{ij}^{(k)}=\lambda p_{ij}^{(l)}$ for any i, j, k, l, where λ is a positive constant, i.e., type A chain in the sense of Doeblin. (B) States $e_1, e_2, \cdots, e_s$ generate one essential class in the sense of Kolmogorov [Bull. Univ. d'Etat Moscou. Ser. Internat. Sect. A. 1 (1937), no. 3, 1–16]. (C) Basic lattice Z coincides with the set of all integral valued vectors of an s-dimensional vector space.

The author proves two limit theorems that are analogous to results for homogeneous Markov chains. The first theorem under conditions (A), (B), and (C) establishes multivariate normality with error term $O[n^{-(s-1)/2}(\ln n)^{-4}]$. The second theorem under conditions (A) and (B) is similar in structure and results. Previously, Kolmogorov [Izv. Akad. Nauk SSSR. Ser. Mat. 13 (1949), 281–300; MR 11, 119] and Siraždinov [Limit theorems for stationary Markov chains, Izdat. Akad. Nauk Uzbek. SSR, Taškent, 1955; MR 18, 944] obtained results for homogeneous chains, and Linnik [Izv. Akad. Nauk SSSR. Ser. Mat. 13 (1949), 553–566; MR 11, 606] proved a local limit theorem for inhomogeneous chains of type A assuming the existence of the "same type of paths."
H. P. Edmundson (Santa Monica, Calif.).

Hammersley, J. M. Markovian walks on crystals. Compositio Math. 11 (1953), 171–186.

The mathematical problem treated here originated from a study of diffusion of electrons in crystals. In the classic theory of diffusion by a random walk, one usually encounters sums of independent random vectors. Such a model can not however be applied directly to a crystal in which there is more than one lattice site in the unit cell because at each instant the set of possible displacements in the walk depends upon the initial position within the cell. The simplest model that one could propose for such a walk would be one in which the displacements depend only upon position within the cell and therefore de-

scribes a Markov process of order 1 over the set of possible displacements.

After reviewing briefly the proofs of some standard theorems, the author proves that if one has a simple Markov process over a finite set of states and if ν_j denotes the number of times the system occupies the jth state in a Markov process of $\nu=\sum \nu_j$ steps, then the distribution of the ν_j is asymptotically joint normal for $\nu\to\infty$ with mean and covariance matrix that can be obtained explicitly from the stochastic matrix. Since a Markov process of finite order is equivalent to one of order 1 and since the states may represent possible displacements in a walk, the theorems can be used to give the sum of vectors selected by a Markov process of order N.

As an example, the author finds the diffusion matrix for a walk on a hexagonal close-packed structure. The analysis of this particular example could however be simplified by making use of some of the symmetry properties of the problem. *G. Newell.*

Miller, Irwin; and Freund, John E. Some results on the analysis of random signals by means of a cut-counting process. J. Appl. Phys. **27** (1956), 1290–1293.

Let $N_T(\theta)$ be the number of times, in the interval $[0, T]$, that a sample function of a stationary Gaussian process with zero means has the value θ. Rice [Bell System Tech. J. **24** (1945), 46–156, p. 60; MR **6**, 233] evaluated $E\{N_T(\theta)\}$. Steinberg, Schultheiss, Wogrin, and Zweig [J. Appl. Phys. **26** (1955), 195–201] found the variance of $N_T(0)$. The authors find an expression for the covariance of $N_T(\theta_1)$, $N_T(\theta_2)$. *J. L. Doob.*

Whittle, P. On the variation of yield variance with plot size. Biometrika **43** (1956), 337–343.

The author is interested in problems in which a random variable $y(\Omega)$ (e.g., the yield of a crop) is associated with each Borel set Ω in a plane (extending agricultural language he calls Ω a "plot"). In practice observation yields information about $V(\Omega)\equiv\operatorname{var}\{y(\Omega)\}$ for various simple forms of Ω and in particular a good deal of information has been accumulated about the way in which $V(\Omega)$ varies with the largest dimension x of Ω when the shape of Ω is fixed. Thus H. Fairfield Smith [J. Agric. Sci. **28** (1938), 1–23] found some evidence for a power-law $V(\Omega)\simeq kx^\nu$ with values of ν in the neighbourhood of $2\frac{1}{2}$–3, and the present author also cites similar situations in the theory of turbulence and in the work of D. R. Cox (unpublished) on variations in yarn diameter.

Whittle now supposes that one can write

$$(*)\qquad V(\Omega)=\iint_\Omega\iint_\Omega \rho(PQ)dPdQ,$$

where

$$\rho(PQ)=\lim\frac{\operatorname{cov}\{y(\Omega_1),\, y(\Omega_2)\}}{|\Omega_1|\cdot|\Omega_2|}.$$

Here Ω_1 and Ω_2 are disjoint neighbourhoods of the distinct points P and Q, with areas $|\Omega_1|$, $|\Omega_2|$, and the limit is taken as these areas contract to zero. The reviewer would have expected to see a term

$$+\iint_\Omega \sigma^2(P)dP$$

on the right-hand side of (*), where

$$\sigma^2(P)\equiv\lim\operatorname{var}\{y(\Omega)\}/|\Omega|$$

and the limit is taken as Ω contracts to P. The omission of this term leads to the conclusion $V(\Omega)\simeq k|\Omega|^2=k_1x^4$ for small plots, while if it had been present the author would have found $V(\Omega)\simeq k|\Omega|=k_1x^2$ for small plots. The omission may be justified in the applications which the author has in mind but it is not commented on in the paper.

For stochastic fields having the property (*) the author then raises the problem of finding the function ρ when V has been determined by experiment; in particular he wants to know what covariance-functions ρ will give power-laws for the dependence of V on the plot-size. He introduces the spectral density function,

$$F(\omega)\equiv\int\exp\{i\omega\cdot x\}\rho(x)dx$$

and the areal characteristic function

$$G(\omega)\equiv\int_\Omega\exp\{i\omega\cdot x\}dx,$$

where x is a coordinate vector and ω is a vector wave-number having as many components as x (two, in the plane case), and so transforms (*) into

$$(\dagger)\qquad V(\Omega)=(2\pi)^{-n}\int|G(\omega)|^2F(\omega)d\omega,$$

where n $(=2)$ is the number of spatial dimensions.

$G(\omega)$ is then evaluated for simple plot shapes (rectangles, circles), when it can be expressed in terms of trigonometric or Bessel functions, and if $V(\Omega)$ is observed for one of these shapes at all possible scales then (†) leads to an integral equation for $F(\cdot)$ with a trigonometric or Bessel kernel. The relation (*) can also be used directly and the author prefers to do this. If the stochastic field is isotropic then $\rho(x)\equiv\rho(r)$, where $r\equiv|x|$, and if for a given plot shape of dimension x we have $V(\Omega)=V_x$ then we shall have

$$(\dagger\dagger)\qquad V_x=x^{2n}\int_0^1 K(s)\rho(xs)\,ds,$$

where $K(\cdot)$ is a function describing the distribution of distances within a plot of the given shape and of unit dimensions. The integral equation (††) is of a form well adapted to solution by the Mellin transform. One plane problem is worked out in complete detail; if the field is isotropic and if V_x is the variance of the random variable associated with a circular plot of radius x, then the author's solution gives $\rho(x)$ in terms of V and its first three derivatives.

The paper concludes with a discussion of the isotropic stochastic fields leading to a power-law for V_x. From what has already been said it will be clear that this pioneer paper is deficient in some details but in the opinion of the reviewer it will open the way to a new and important development of stochastic process theory.

D. G. Kendall (Oxford).

Faure, Pierre. Sur quelques résultats relatifs aux fonctions aléatoires stationnaires isotropes introduites dans l'étude expérimentale de certains phénomènes de fluctuations. C. R. Acad. Sci. Paris **244** (1957), 842–844.

On étudie diverses propriétés de fonctions aléatoires stationnaires isotropes F introduites par l'étude de le transparence de films photographiques uniformément impressionnés; on généralise ces propriétés au cas de fonctions aléatoires F définies sur un espace euclidien E_n à n dimensions. On insiste particulièrement sur les rapports du spectre de F avec ceux de la trace de F sur des sous-espaces de E_n. *Author's summary.*

Skitovič, V. P. On characterizing Brownian motion. Teor. Veroyatnost. i Primenen. 1 (1956), 361–364. (Russian. English summary)

Let $X(t)$ be a process with stationary independent increments; the author proves the following theorem. If $a(t)$ and $b(t)$ are continuous functions of t on the closed interval $[a, b]$, if $\int_a^b [a(t)b(t)]^2 dt \neq 0$, if one of the integrals $\int_a^b a(t)^2/b(t)^2\, dt$, $\int_a^b b(t)^2/a(t)^2\, dt$ exists, and if the stochastic integrals

$$Y=\int_a^b a(t)dX(t),\ Z=\int_a^b b(t)dX(t)$$

are independent, then $X(t)$ is a Brownian movement process.

The stochastic integrals X and Y are defined as limits in distribution of the approximating sums (in fact, the distributions of the sums are required to converge to a limit distribution uniformly over the entire real line), and it is shown that such a limit always exists if the integrand is continuous and the proces $X(t)$ has stationary independent increments.

An analogous theorem for sequences of independent random variables is also stated.

E. J. Kelly

See also: Kampé de Fériet, p. 40; Kruskal, p. 67; Thomson and Barton, p. 77; Siegel, p. 79; Saaty, p. 106; Burke and Estes, p. 106.

STATISTICS

★**Steinhaus, Hugo. Über einige prinzipielle Fragen der mathematischen Statistik.** Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober, 1954, pp. 55–63. Deutscher Verlag der Wissenschaften, Berlin, 1956.

The author discusses some difficulties in the foundations of mathematical statistics, "welche über die von der Routine gezeichnete Grenzlinie hinaus zu Problemen von prinzipieller Bedeutung führen." Laplacian determinism and the Steinhaus-Kac theory of independent functions are accepted in statistical mechanics. The author's remark that one example of compatibility suffices to dissipate the "antinomy" between deterministic and probabilistic mechanics might be considered questionable by logicians. He accepts the Bayes-Laplace hypothesis of homogeneous a priori distributions without mentioning the contradictions to which it leads. As a measure of "amount of information" the quantity $[\int f^2(t)dt]^{\frac{1}{2}}$ is used, $f(t)$ being the probability density, without mention of the customary definitions using $\log f(t)$ and their main advantage: additivity under independence. The problem of inductive inference is reduced in a short passus to a question on Darwinism and conditioned reflexes.

D. van Dantzig (Amsterdam).

★**Hemelrijk, J.; en Wabeke, Doraline. Elementaire statische opgaven met uitgewerkte oplossingen. [Elementary problems in statistics with their solutions.]** J. Noorduijn en Zoon N.V., Gorinchem, 1957. 154 pp.

This is volume 2 of the Centrumreeks published by the Matematisch Centrum in Amsterdam.

Sakoda, James M.; and Cohen, Burton H. Exact probabilities for contingency tables using binomial coefficients. Psychometrika 22 (1957), 83–86.

The use of binomial coefficients in place of factorials to shorten the calculation of exact probabilities for 2×2 and $2\times r$ contingency tables is discussed. A useful set of inequalities for estimating the cumulative probabilities in the tail of the distribution from the probability of a single table is given. A table of binomial coefficients with four significant places and n through 60 is provided.

Authors' summary.

Bulmer, M. G. Approximate confidence limits for components of variance. Biometrika 44 (1957), 159–167.

Suppose that M_1 and M_2 are independent mean-square variates with f_1 and f_2 degrees of freedom and unknown expected values $(\theta+\sigma^2)$ and σ^2 respectively. The author considers the problem of finding a function $f(M_1, M_2)$ such that approximately $\Pr[f(M_1, M_2)\leqq\theta]=\alpha$, whatever θ and σ^2 are; that is, he seeks "approximate confidence limits" for θ. As an approximate solution, he suggests the function $M_2g_2(F)$, where

$$g_2(F)=FL_2^{-1}-1+L_1F^{-1}(1-L_1L_2^{-1});$$

here $F=M_1/M_2$, and L_1 and L_2 are respectively the lower $100\alpha\%$ point for the F-distribution with f_1 and f_2 degrees of freedom and the lower $100\alpha\%$ point for the F-distribution with f_1 and ∞ degrees of freedom. If $\Pr[M_2g_2(F)\leqq\theta]$ is set equal to P, then the error $(P-\alpha)$ is of order $f_1^{2}f_2^{-2}$. The paper concludes with numerical comparison of the results given by the present approximation with the results of the approximations of Welch [J. Amer. Statist. Assoc. 51 (1956), 132–148; MR 17, 1103] and Huitson [Biometrika 42 (1955), 471–479; MR 17, 279].

R. G. Stanton (Waterloo, Ont.).

Lal, D. N.; and Mishra, D. Distribution of the ratio of the logarithm of any one of the ranges of samples from a rectangular population to the sum of the logarithms of each of them. J. Indian Soc. Agric. Statist. 7 (1955), 179–186.

The distribution of the ratio of the logarithm of the range of any one of K independent random samples from a continuous rectangular population, to the sum of their logarithms, is derived. The probability that at least the largest of the ratios exceeds a given number W is discussed and a small numerical table giving the values of this probability for various selected values of W is given.

J. H. Curtiss (Providence, R.I.).

Bennett, B. M. Certain multivariate distributions in the presence of intraclass correlation. J. Indian Soc. Agric. Statist. 7 (1955), 70–72.

This note gives a brief exposition of the effects on the Wishart distribution, and on the distribution of Hotelling's T, of intraclass correlation between observations from a multivariate normal population. *J. H. Curtiss.*

Good, I. J. On the serial test for random sequences. Ann. Math. Statist. 28 (1957), 262–264.

A finite random sequence of length N is to be tested for independence of the trials, where the variates each may assume any one of t values. The number of each kind of subsequence of fixed length ν is counted, and a statistic analogous to chi-square is calculated. The paper proves that this statistic has an expected value of $t^\nu-1$, provided the trials are independent and that $\nu\leqq\frac{1}{2}(N+1)$. The same result holds when circular sequences are considered.

S. W. Nash (Vancouver, B.C.).

Noether, Gottfried E. Two confidence intervals for the ratio of two probabilities and some measures of effectiveness. J. Amer. Statist. Assoc. **52** (1957), 36–45.

Confidence limits for the quantities $(p_2-p_1)/(1-p_1)$ and $(p_1-p_2)/p_1$, where p_1 and p_2 are the parameters of two binomial distributions, are derived by using the normal approximation and some of the known asymptotic formulas for the distributions of ratios involving normal random variables. *J. H. Curtiss* (Providence, R.I.).

Mitra, Sujit Kumar. Tables for tolerance limits for a normal population based on sample mean and range or mean range. J. Amer. Statist. Assoc. **52** (1957), 88–94.

Let $\bar{x}$ and r be respectively the observed mean and range in a random sample of size n from a normal population. Values of k_1 are tabulated which ensure a probability β that the pair of limits $\bar{x}\pm k_1r$ will include at least a proportion p of the population. For tolerance limits of the form $\bar{\bar{x}}\pm k_2\bar{r}$, where $\bar{\bar{x}}$ is the grand mean and $\bar{r}$ the mean range in N samples of size m each, necessary tables of the factor k_2 are provided for situations where m is either 4 or 5, these being common group sizes in control chart analysis. *Author's summary.*

Wold, H.; and Faxér, P. On the specification error in regression analysis. Ann. Math. Statist. **28** (1957), 265–267.

Given the theoretical relation $y=\beta_1x_1+\cdots+\beta_hx_h+\zeta$, suppose (a) the disturbance ζ has zero expectation and finite variance $\sigma^2(\zeta)$, and (b) none of the explanatory variables $x_1\cdots x_h$ is identically linear in the other ones. Let $y=b_1x_1+\cdots+b_hx_h+z$ be the least squares regression of y on $x_1\cdots x_h$. Then $|b_i-\beta_i|\leq\sigma(\zeta)/\sigma(x_i)\cdot(1-R_i^2)^{\frac{1}{2}}$, where R_i is the multiple correlation coefficient of x_i and $x_1\cdots x_{i-1}x_{i+1}\cdots x_h$. Assume (A) $\sigma(\zeta)\leqq\varepsilon\cdot\sigma(x_i)$ $(\varepsilon\geqq0)$. (B) Consider the boundary of the ellipsoid:

$$\begin{vmatrix}\rho_{ij} & u_i\\ u_j & 1\end{vmatrix}=0\quad(i,j=1,\cdots,h),$$

where ρ_{ij} is the simple correlation coefficient of x_i and x_j; $u_1\cdots u_h$ are coordinates in h-dimensional Euclidean space. Let $r^*_1\cdots r^*_h$ be on the boundary. Let r_i be the simple correlation coefficient of x_i and ζ. Assume $r_i=\varepsilon'\cdot r^*_i$ $(0\leqq\varepsilon'\leqq1)$. Then $|b_i-\beta_i|\leqq\varepsilon\cdot\varepsilon'/(1-R_i^2)^{\frac{1}{2}}$. Hence if ε and ε' are small, the specification error of the regression coefficient b_i will be at most of order $\varepsilon\cdot\varepsilon'$. This theorem is related to Wold's proximity theorem [H. Wold and L. Juréen, Demand analysis, Wiley, New York, 1953, pp. 192 ff; MR 16, 274]. *G. Tintner* (Ames, Iowa.).

Basmann, R. L. A generalized classical method of linear estimation of coefficients in a structural equation. Econometrica **25** (1957), 77–83.

An independent review of the formal relations between least squares estimation and the specific econometric approaches [cf. H. Theil, Bull. Inst. Internat. Statist. **24** (1954), 2ème livraison, 122–129; MR 16, 1040]. *H. Wold* (Uppsala).

Bennett, B. M. On confidence limits for the ratio of regression coefficients. Ann. Inst. Statist. Math., Tokyo **8** (1956), 41–43.

Using Barankin's t-statistic [Proc. Berkeley Symposium Math. Statist. Probability, 1945, 1946, Univ. of California Press, 1949; pp. 433–449; MR **10**, 467] and Fieller's theorem [Quart. J. Pharmacy and Pharmacology **17** (1944), 117–123] the author obtains a confidence interval for the ratio of the regression coefficients of two bivariate normal populations with unknown and possibly unequal residual variances. The sizes of the corresponding samples are not assumed equal. {Reviewer's remark. For recent comments on Fieller's method from the point of view of the theory of confidence intervals see Neyman, J. Roy. Statist. Soc. Ser. B. **16** (1954), 216–218}. *D. M. Sandelius* (Göteborg).

Zelen, Marvin. The analysis of incomplete block designs. J. Amer. Statist. Assoc. **52** (1957), 204–217.

The author considers first tests of hypotheses utilizing intra- and interblock information separately. A method is proposed for combining the intra- and interblock tests and the gain in power over the intrablock test is computed for an example. Conditions are stated under which all treatment contrasts may be tested by the proposed method and these conditions seem to be often met in practice. The author also gives a confidence interval for the ratio of intrablock to interblock variance.

It remains an open question how the author's method compares in power with the customary methods of utilizing interblock information. *H. B. Mann.*

Seal, K. C. A note on sums of covariances of order statistics from normal populations. Calcutta Statist. Assoc. Bull. **7** (1956), 33–34.

Suppose that there are n normal populations with different means and a common variance, and that n random observations are drawn, one from each population. Let $X_{(1)}<X_{(2)}<\cdots<X_{(n)}$ be the n order statistics obtained by ordering the sample. The author presents a new proof of the fact that $\sum_{j=1}^n\sigma_{ij}=1$, $i=1,2,\cdots,n$, where $\sigma_{ij}=\mathrm{cov}\,(X_{(i)},X_{(j)})$. This result was derived earlier by the author [Ann. Math. Statist. **27** (1956), 854–855; MR **18**, 159]. He achieves the present proof by showing that for $p\neq q=1,\cdots,n$, $X_{(p)}-X_{(q)}$ is stochastically independent of $\sum_1^n X_{(i)}$, a fact which he refers to a theorem of Basu and Laha [Sankhyā **13** (1954), 359–362; **14** (1954), 180; MR **16**, 51]. *J. H. Curtiss* (Providence, R.I.).

Adler, Leta McKinney. A modification of Kendall's tau for the case of arbitrary ties in both rankings. J. Amer. Statist. Assoc. **52** (1957), 33–35.

The sample analogue of a variant of Kendall's τ is suggested for cases in which ties appear in both rankings of a bivariate sample. The variant considered was proposed by L. A. Goodman and the reviewer [J. Amer. Statist. Assoc. **49** (1954), 732–764], and its sample analogue was described by W. A. Wallis and H. V. Roberts [Statistics: a new approach, Free Press, Chicago, 1956]. A comparison is made with a statistic proposed by Kendall for the tied case. In the displayed expression following (3), p. 34, a left parenthesis is missing before the final denominator T_1. *W. Kruskal* (Chicago, Ill.).

Cartwright, Desmond S. A computational procedure for tau correlation. Psychometrika **22** (1957), 97–104.

The tau coefficient is defined, and a computational procedure for tied ranks is described. *From the author's summary.*

Chakravarti, I. M. Fractional replication in asymmetrical factorial designs and partially balanced arrays. Sankhyā **17** (1956), 143–164.

The author considers factorial experiments where the

factors can be divided into g groups such that the ith group contains m_i factors each of which is on s_i levels. Given orthogonal arrays $(N_i, m_i, s_i, d_i+k_i-1, \lambda_i)$, $i=1, \cdots, g$, the author constructs from them a fractional replication of the factorial design from which all main effects and certain interactions are estimable under the assumption that certain higher order interactions are 0. In certain special cases two other methods devised by the author allow further reductions in the number of experiments. Construction and analysis of these designs is illustrated by examples. *H. B. Mann* (Columbus, Ohio).

Baker, Anthony G. Analysis and presentation of the results of factorial experiments. Appl. Statist. **6** (1957), 45–55.

In this article the author describes a systematic method of setting down and analysing the results of factorial experiments. The method is particularly suitable for use with a desk calculating machine and is applicable to either qualitative or quantitative factors.

Author's summary.

Box, G. E. P.; and Hunter, J. S. Multi-factor experimental designs for exploring response surfaces. Ann. Math. Statist. **28** (1957), 195–241.

Suppose a relationship $\eta=\phi(\xi_1, \xi_2, \cdots, \xi_k)$ exists between η and the levels $\xi_1, \xi_2, \cdots, \xi_k$ of k quantitative factors, such that within a limited region of interest in the space of the variables, it can be adequately represented by a polynomial of degree d. The authors define a k-dimensional experiment design of order d as a set of N points in the k-dimensional space of the variables such that the data generated enables all the coefficients in the dth degree polynomial to be estimated. Let $\hat{y}_x$ be the estimated response at the point $x=(x_1, x_2, \cdots, x_k)$ in the space of the variables using a polynomial fitted by least squares to N observations made in accordance with some experiment design. Then $NV(\hat{y}_x)/\sigma^2$ is a standardized measure of the precision with which the design allows the response at x to be estimated. The authors define rotatable designs as designs such that $V(\hat{y}_x)$ is a function of $\rho^2=\sum_{i=1}^k x_i^2$ in the $\{x_i\}$. Conditions are found for designs to be rotatable and examples are given of rotatable designs of order 1 and order 2. Conditions for blocking are derived and examples given. Confidence regions for the maximum of a second degree equation is also discussed when a rotatable design is used to generate the data. *M. Zelen* (Washington, D.C.).

Holloway, Clark, Jr. A systematic method of finding defining contrasts. J. Amer. Statist. Assoc. **52** (1957), 46–52.

When a 2^N factorial design is fractionated, and only 2^{N-p} treatment combinations observed, all of the confounded effects are fixed by the selection of the set of defining contrasts. Of the 2^p elements in this set, the identity is always included; but only p independent elements may be chosen arbitrarily. The author develops a systematic method for finding the sets of defining contrasts which satisfy the following criterion: all main effects plus as many as possible of the two factor interactions must be confounded only with high-order interactions. The method consists of writing out a matrix whose columns are the single degree of freedom coefficients with 1 identified as X and -1 as a blank or zero, and the rows represent the treatment combinations of a 2^p design. The author then gives rules for assigning each of the letters associated with the 2^N design to the elements of a column of the matrix to yield a set of defining contrasts. *M. E. Terry* (Murray Hill, N.J.).

Shirafuji, Michie. Note on the determination of the replication numbers for the slippage problem in r-way layout. Bull. Math. Statist. **7** (1956), 46–51.

The author considers the problem of finding that combination of levels $a_1, \cdots, a_\alpha$ which will maximize a given interaction $\mu(i_1, \cdots, i_\alpha; a_1, \cdots, a_\alpha)$. (The notation is that of the reviewer's book.) This is done by choosing that combination of levels $a_1, \cdots, a_\alpha$ for which the observed value of $x(i_1, \cdots, i_\alpha; a_1, \cdots, a_\alpha)$ is largest. This procedure was justified by R. R. Bahadur [Ann. Math. Statist. **21** (1950), 362–375; MR **12**, 117].

Following this procedure and assuming a reasonable form of the risk function, the author then proceeds to compute the sample size which will minimize the maximum risk. The solutions for the sample size when main effects and first order interactions are to be maximized are explicitly given in the paper. The author states that the solutions for higher order interactions have also been obtained but that they are very complicated. The derivation is given in detail for main effects and indicated for first order interactions.

It seems to the reviewer that the combination of levels selected will tend to maximize $E(x(i_1, \cdots, i_\alpha; a_1, \cdots, a_\alpha))$ rather than $\mu(i_1, \cdots, i_\alpha; a_1, \cdots, a_\alpha)$ as stated by the author. This fact does not affect the practical significance of the result because $E(x(i_1, \cdots, i_\alpha; a_1, \cdots, a_\alpha))$ is precisely the quantity one would wish to maximize in a practical situation. *H. B. Mann* (Columbus, Ohio).

★ **Cochran, William G.; and Cox, Gertrude M. Experimental designs. 2nd ed.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London. 1957. xiv+617 pp. $10.25.

The first edition was reviewed in MR **11**, 607. In the present edition two chapters have been added dealing with factorial experiments. Sections have also been added on incomplete block designs. There are various other briefer additions.

★ **Bozivich, Helen; Bancroft, T. A.; Hartley, H. O.; and Huntsberger, David V. Analysis of variance: preliminary tests, pooling, and linear models.** WADC Tech. Rep. 55-244, vol 1. Wright Air Devel. Center, Wright-Patterson Air Force Base, Ohio, 1956. vi+139 pp.

Part I discusses certain pooling procedures with regard to the power and size resulting from them on the basis of certain incompletely specified linear models. The main results and techniques are identical with those of the paper published previously by the first three authors [Ann. Math. Statist. **27** (1956), 1017–1043; MR **18**, 608]. However this monograph contains more extensive tables and figures of size curves and power curves with varying degrees of freedom and some practical illustrations of the uses of pooling procedure.

Part II by D. H. Huntsberger describes and illustrates a generalised pooling procedure which utilizes a weighted estimator. The efficiencies of the weighting procedure, for a particular weighting function, and of the sometimes-pool procedures are compared relative to the never-pool procedure in the case of normally distributed estimators. Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two independent, unbiased, normally distributed estimators for θ_1 and θ_2 respectively. Let $\hat{\theta}_1$

and $\hat{\theta}_2$ have known variances σ_1^2 and σ_2^2 respectively. A pooled estimator for θ_1 is obtained as

$$W(T)=\phi(T)\hat{\theta}_1+[1-\phi(T)]\frac{\sigma_2^2\hat{\theta}_1+\sigma_1^2\hat{\theta}_2}{\sigma_1^2+\sigma_2^2},$$

where $\phi(T)$ is a function of T only and

$$T=(\hat{\theta}_1-\hat{\theta}_2)/(\sigma_1^2+\sigma_2^2)^{\frac{1}{2}}.$$

The three special cases $\phi_0(T)=T^2(1+T^2)^{-1}$, $\phi_{ab}(T)=1-ae^{-bT^2}$ and $\phi_{sp}(T)=0$ or 1 according as $|T|<$ or $\geqq t_\alpha$, where $P(|t|\geqq t_\alpha)=\alpha$ and t is the standard normal deviate, are compared with each other. It is shown that for the case $\phi_0(T)$ the weighting procedure has greater control over possible disturbances resulting from pooling than does the sometimes-pool procedure. *T. Kitagawa.*

Sprott, D. A. Correction to "A note on combined inter-block and intrablock estimation in incomplete block designs". Ann. Math. Statist. **28** (1957), 269.

The third sentence of the original paper, reviewed in MR **18**, 244, is now deleted.

Armitage, P. Restricted sequential procedures. Biometrika **44** (1957), 9–26.

Let $\{x_i\}$ $(i=1, 2, 3, \cdots)$ be NID (μ, σ^2) with μ known, σ unknown; and let $y_n=\sum_{i=1}^n x_i$. Restricted sampling procedures are considered which consist of sampling until $y_n\geqq a+bn$ or $y_n\leqq -a-bn$ or $n=N$. The constants a, b, n are required to satisfy the restrictions

$$\Pr[y_n\geqq a+bn|\ \mu=0]=\Pr[y_n\leqq -a-bn|\ \mu=0]=\alpha,$$

$$\Pr[y_n\geqq a+bn|\ \mu=\mu_1>0]=1-\beta,$$

with α, β preassigned.

The author studies one member of the family of such restricted sequential procedures; to calculate the required probabilities he approximates the discrete random walk by a continuous diffusion process. In particular with this approximation an equation relating N and β is found. He also gives a similar restricted sequential procedure to test the difference between the means of binomial distributions, based on the same approximation. The exact probabilities calculated by an enumeration of paths are compared with the approximate probabilities for $N=44$.

D. G. Chapman (Seattle, Wash.).

Petrov, A. A. Verification of statistical hypotheses on the type of a distribution based on small samples. Teor. Veroyatnost. i Primenen. **1** (1956), 248–271. (Russian. English summary)

Tulse, R. Sampling for variables with a very skew distribution. Appl. Statist. **6** (1957), 40–44.

This article propounds a sampling method that may be useful in certain types of survey work, and describes a test of its efficiency when applied to an artificial population with known characteristics. *Author's summary.*

Stannage, William. Use of regression analysis to detect errors in measurement of intermediate materials in a multi-stage process. Appl. Statist. **6** (1957), 63–66.

The author shows how certain unexpectedly wide variations in plant efficiency were found to be largely due to errors of measurement. *Author's summary.*

Longuet-Higgins, M. S. A statistical distribution arising in the study of the ionosphere. Proc. Phys. Soc. Sect. B. **70** (1957), 559–565.

An approximate distribution is deduced for the directions of the 'lines of maxima' on a random surface. The theoretical distribution is found to be in good agreement with some experimental results obtained previously by Briggs and Page. *Author's summary.*

Moore, P. G. Sampling techniques and some applications. J. Inst. Actuar. Students' Soc. **14** (1957), 111–128.

Expository paper with applications primarily from the actuarial field. *D. G. Chapman* (Seattle, Wash.).

Johnson, N. L.; and Moore, P. G. Applications of sequential methods to mortality data. J. Inst. Actuar. Student's Soc. **14** (1957), 84–93.

Some examples of applications of sequential analysis to questions which arise in actuarial work are presented. Typically the questions center about whether or not a standard mortality table is in accord with current experience. *J. H. Curtiss* (Providence, R.I.).

See also: Steinhaus, p. 73; Wendel, p. 58; Barton and David, p. 70; Burkholder, p. 71; Davin, p. 77; Rosenblatt, p. 87; Frahn, p. 102; Ja'nossy and Kiss, p. 103; Seng, p. 105; Biser and Meyerson, p. 106.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

Kalicin, Nikola St. On a new mechanics of the nucleus. Jbuch. Staatsuniv. Stadt Stalin Fak. Bauwesen **1** (1953), 151–180. (Bulgarian. Russian summary)

A five-dimensional space is introduced with two time-like coordinates, so that a particle in its motion describes a surface (instead of a curve). Instead of the interval, the element of area is taken as the fundamental geometrical entity. A formalism is developed leading to a new kind of mechanics. *N. Rosen* (Haifa).

★ **Banach, Stefan. Mechanika. [Mechanics.] 4th ed., revised. Biblioteka Matematyczna. Tom. 13.** Państwowe Wydawnictwo Naukowe, Warszawa, 1956. 558 pp. zł. 32.

An English translation was reviewed in MR **13**, 290.

Rosenberg, R. M. A pursuit problem. J. Franklin Inst. **262** (1956), 265–279.

The problem solved here is a generalization of the missile pursuit curve with lead, generalized in the sense that (a) the (constant) altitude of the game (target) is unknown, and (b) the vision of the airborne hunter (missile) is restricted to a range linearly dependent upon his altitude — he is blind at greater or lesser ranges. The game is assumed to be moving in a fixed direction at constant speed, the hunter moves at a fixed speed in some trajectory to be determined so that at the instant the hunter sights the game, he can begin moving (blindly, of course) along the tangent, at his constant speed, and intercept the game.

The three-dimensional kinematic equations describing the motion are easily set up, and an ingenious transformation, introducing a set of non-orthogonal coordinates, simplifies the problem to the solution of the first order

differential equation

$$d\lambda/d\mu = \sqrt{((\lambda/\mu)^2 - 1)},$$

from which the hunter's trajectory is found.
R. E. Gaskell (Seattle, Wash.).

Rosenauer, N. Some fundamentals of space mechanisms. Math. Gaz. **40** (1956), 256–259.
Examples of n-link space mechanisms, $n=3, 4, 5, 6, 7$, the joints having various degrees of freedom. Clearly drawn figures are added. *O. Bottema* (Delft).

Glagolev, A. A. Application of the theory of generalized throws to the kinematics of collinearly changing systems. Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. **21** (1954), 83–90. (Russian)

Hain, K. Diagonalwinkel-Zuordnungen in Gelenkviereck. Ing.-Arch. **25** (1957), 193–200.

Meyer zur Capellen, W. Die Extrema der Geschwindigkeiten an Kurbeltrieben. Ing.-Arch. **25** (1957), 140–154.

Parkyn, D. G. The inverting top. Math. Gaz. **40** (1956), 260–265.
This paper deals with the motion of the tippe-top [cf. C. M. Braams, Physica **18** (1952), 503–514; MR **14**, 421; N. M. Hugenholtz, ibid. **18** (1952), 515–527; MR **14**, 421; J. L. Synge, Philos. Mag. (7) **43** (1952), 724–728; MR **14**, 100]. The plan is to regard sliding friction as a perturbation on a frictionless steady precession, the fast precession being selected rather than the slow one. With approximations based on the fastness of the spin, the author finds that the axis falls if

$$\frac{h}{a} + \frac{A-C}{C} \cos\alpha > 0,$$

where h=height of geometrical centre of base above mass-centre, a=radius of base, A, C=transverse and axial moments of inertia, α=inclination of axis to vertical. The ultimate rising of the top on its peg is also discussed. [Since h and $A-C$ are both positive in the actual tippe-top, the above inequality is satisfied for all acute angles α. This indicates, in particular, instability when the axis is vertical ($\alpha=0$) and the spin is sufficiently great, a result in agreement with O'Brien and Synge, Proc. Roy. Irish Acad. Sect. A. **56** (1954), 23–35; MR **15**, 659].
J. L. Synge (Dublin).

★ **Кориолис, Г. [Koriolis, G.] Математическая теория явлений бильярдной игры. [Mathematical theory of the phenomena of billiard games.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 235 pp. 5.6 rubles.
A translation by I. N. Veselovskiĭ and M. M. Gernet from the French edition originally published in 1835. The introduction to the Russian edition contains a short biography of Coriolis.

★ **Masotti, Arnaldo. Sopra una estensione di un teorema di Newton relativo ai moti centrali parabolici.** Scritti matematici in onore di Filippo Sibirani, pp. 167–172. Cesare Zuffi, Bologna, 1957.
A study of the possibilities for central motion along a parabola when the center of force does not coincide with the focus of the parabola.

Dobronravov, V. V. Lagrange's theorem in nonholonomic coordinates. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **10** (1953), no. 2, 186–190. (Russian)

Godfrey, D. E. R. Two-dimensional stress and inertia combinations. J. Roy. Aero. Soc. **61** (1957), 353–354.

Manžeron, D. I. On induced accelerations of arbitrary order and some of their extremal properties. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 27–28. (Russian)

Nikitin, A. K. Singularities of the canonical equations of dynamics. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. **18** (1953), no. 3, 43–48. (Russian)

Rau, P. S. Kinematics of a rigid body. Math. Student **24** (1956) 226–229 (1957).
A rederivation of the Coriolis theorem.

Skimel', V. N. On problems of stability of motion of a heavy rigid body about a fixed point. Prikl. Mat. Meh. **20** (1956), 130–132. (Russian)

Šulgin, M. F. On analytic dynamics in quasicoordinates. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **10** (1953), no. 2, 191–195. (Russian)

Zunnunov, N. Z. On the equations of motion of nonholonomic systems with linear restraints. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **13** (1954), 121–128. (Russian)

Thomson, W. T.; and Barton, M. V. The response of mechanical systems to random excitation. J. Appl. Mech. **24** (1957), 248–251.
This expository paper calculates the mean square response of a lightly damped, multi-degree of freedom system to a random input of prescribed power spectral density. The general results are applied to the motion of rods and beams. *J. W. Miles* (Los Angeles, Calif.).

Davin, M. Etudes statistiques sur la résistance des corps prismatiques soumis à des champs de contrainte uniformes. Ann. Ponts Chaussées **126** (1956), 719–754, 127 (1957), 1–18.

Chadenson, L. Essais sur les théories aérodynamiques relatifs aux ponts suspendus et leur application au pont de Tancarville. Ann. Ponts Chaussées **127** (1957), 19–87, 169–225.

Halimanovič, M. P. On the motion of a not completely symmetric heavy gyroscope for small angles of nutation. Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. **15** (1953), 61–67. (Russian)

Halimanovič, M. P. On a generalization of the notion of coefficient of gyroscopic stability. Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. **19** (1954), 93–98. (Russian)

Harlamov, P. V. Integrable cases in the problem of motion of a heavy solid body in a fluid. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 381–383. (Russian)

Nikitin, A. K. On stability of steady motion of a canonical system. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. **18** (1953), no. 3, 49–54. (Russian)

Rumyancev, V. V. Stability of permanent rotations of a heavy rigid body. Prikl. Mat. Meh. **20** (1956), 51–66. (Russian)

Culler, Glen J.; and Fried, Burton D. Universal gravity turn trajectories. J. Appl. Phys. **28** (1957), 672–676.

Maravall Casenoves, Darío. On the dynamics of systems with variable mass. Gac. Mat., Madrid (1) **8** (1956), 256–262. (Spanish)

The author considers a dynamical system consisting of n particles subject to constraints and forces. It is assumed that the masses of the particles are of the forms $m_1=a_1m(t), \cdots, m_n=a_nm(t)$, where the a's are constants, and that the constraints and forces may also depend explicitly on the time. It is shown that by a transformation, $\tau=\tau(t)$, of the independent variable the differential equations of motion can be reduced to the form which applies in the case in which the masses are constants. Conditions are found under which the Hamilton-Jacobi equation for the system can be reduced, by a separation of variables procedure, to a form in which the new time variable τ is not involved. These conditions are satisfied in various simple cases, and some of these cases are discussed in detail. *L. A. MacColl* (New York, N.Y.).

Sapa, V. A. Variational principles in the mechanics of variable mass. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. **1956**, no. 5(9), 116–125. (Russian)

The paper deals with the variational and non-variational principles in the mechanics of variable mass. First, the differential principles are treated (the principle of the virtual displacements, d'Alembert's principle and the least curvature principle of Gauss). Lagrange's equations of motion of the 1st kind are deduced from Gauss's principle and inversely, and this principle from d'Alembert-Lagrange's general differential equations of motion. Further, Hamilton's principle is treated and from it Lagrange's differential equations of the 2nd kind are deduced. The principles are applied to systems of variable mass rotating about a fixed axis. The author observes that the principle of least action can not be deduced because the integral of energy does not exist for such systems. *D. Rašković* (Belgrade).

See also: La Salle, p. 35; Pell, p. 67; Sved, p. 94; Kadymov, p. 107; Cremona, p. 108.

Statistical Thermodynamics and Mechanics

Hijmans, J.; and de Boer, J. An approximation method for order-disorder problems. IV. Physica **22** (1956), 408–428.

Hijmans, J. An approximation method for order-disorder problems. V. Physica **22** (1956), 429–442.

These papers present a new and simpler derivation of the order-disorder procedure given in previous papers of the same authors [Physica **21** (1955), 471–484, 485–498, 499–516, MR **17**, 930]. A "lattice gas" is introduced and the derivation is based on the free energy expressions using the "fugacity" of J. E. Mayer. A variety of special probability quantities are introduced and manipulated in order to permit a relatively broad approach to order-disorder. Numerical results are not given in these papers. *F. J. Murray* (New York, N.Y.).

Van Hove, Léon. Statistical mechanics: a survey of recent lines of investigation. Rev. Mod. Phys. **29** (1957), 200–204.

In this survey the following problems of statistical mechanics are — necessarily briefly — reviewed. Application of perturbation theory to crystals and dilute gases; liquids and dense gases; condensation of gases and ferromagnetism; irreversible processes and the approach to equilibrium. *N. G. van Kampen* (Utrecht).

Cohen, E. G. D.; De Boer, J.; and Salsburg, Z. W. A cell-cluster theory for the liquid state. III. The harmonic oscillator model. Physica **23** (1957), 389–403.

In this paper the cell-cluster theory for the liquid state developed in previous papers [see the following review] is applied to the case that the total intermolecular potential field can be approximated by a harmonic potential field.

This model can be applied to the solid state as well as to the liquid state and gives an improvement on the single-cell (Einstein) theory of the condensed phase in the direction of the results of an exact treatment. Already the first cell-cluster approximation accounts for about 50% of the entropy difference between the single-cell and the exact theory. *Authors' summary.*

Salsburg, Z. W.; Cohen, E. G. D.; Rethmeier, B. C.; and De Boer, J. Cell-cluster theory of the liquid state. IV. A fluid of hard spheres. Physica **23** (1957), 407–422.

The cell-cluster theory of the liquid state, developed in two previous papers [M. N. Rosenbluth and A. W. Rosenbluth, J. Chem. Phys. **22** (1954), 881–884; J. De Boer, Proc. Internat. Confer. Theoret. Phys., Kyoto and Tokyo, 1953, Sci. Council Japan, Tokyo, 1954, pp. 507–530; Physica **20** (1954), 655–664; MR **16**, 658], is applied here to a fluid of hard elastic spheres. The corrections introduced by considering cell-clusters of two cells are investigated. The double cell partition function is calculated in an approximate way and the influence on the entropy and the equation of state is studied. The results of the cell-cluster theory for the equation of state are compared with the recent Monte-Carlo calculations for a hard sphere fluid by Rosenbluth and Rosenbluth [E. G. D. Cohen, J. De Boer, and Z. W. Salsburg, ibid. **21** (1955), 137–147]. *Authors' summary.*

Shabanskii, V. P. Transfer processes in conductors with regard to nonlinear effects. Soviet Physics. JETP **4** (1957), 497–508.

In a previous paper [Ž. Eksper. Teoret. Fiz. **27** (1954), 142–146] Shabanskii derived kinetic equations allowing for the possible heating of an electron gas relative to the lattice. But one may expect deviations from Ohm's law and nonlinearities in other effects, on account of the heating of the electrons. Therefore Shabanskii investigates all combinations of transfer processes in conductors with regard to the nonlinear effects and heating of the electrons. At first, using the tool of statistical mechanics, he considers the relaxation processes in a metal and the temperature of the electrons in an electric field. The changes in the electron distribution function are due to electron-phonon, electron-electron and electron-impurity collisions and those in the phonon distribution function are due to phonon-electron, phonon-phonon and phonon-impurity collisions. The heating of electrons in transfer processes can be calculated with the aid of the kinetic equation, which integrated over the phase space furnishes the equation of conservation of charge and integrated with respect to the energy of the electron furnishes the equation of conservation of energy. Solutions of kinetic equations in two extreme cases: strong and small interelec-

tronic interaction, furnish the kinetic coefficients in the transfer equations. Using these, Shabanskii investigates galvanomagnetic and thermoelectric phenomena: electric conductivity, resistance changes in a magnetic field, the Thomson and Peltier coefficients and the resistance of a polycrystal. *M. Z. Krzywoblocki* (Urbana, Ill.).

Iwamoto, Fumiaki; and Yamada, Masami. Cluster development method in the quantum mechanics of many particle system. I. Progr. Theoret. Phys. **17** (1957), 543–555.

A direct approach to the many-body problem with strong forces leads to considerable difficulties. In an attempt to overcome these difficulties Jastrow [Phys. Rev. (2) **98** (1955), 1479–1484] used the cluster development method from the classical theory of an imperfect gas. The present paper is the first of a series where this method is developed in a more systematic way. A variational procedure is used. Two-particle functions are included in the trial functions to represent the correlations between particles. The energy expectation value is expanded into cluster integrals. The applicability of the method is so far restricted to cases where this expansion converges rapidly. Long range correlations cannot be taken into account. Both fermion and boson systems are considered. *E. Gora* (Providence, R.I.).

Gubanov, A. I. Scattering of electrons in a liquid due to violation of long range order. Soviet Physics. JETP 3 (1957), 854–861.

Earlier papers of the author develop an approximation for the motion of electrons in liquid metals based on a coördinate transformation such that the lattice becomes periodic. Here he studies the corrections to this theory, and in particular the extra scattering caused by the terms introduced by this transformation — the so-called "liquid scattering". Various qualitative results are obtained. *P. W. Anderson* (Murray Hill, N.J.).

Temperley, H. N. V. The statistical mechanics of the steady state. Proc. Phys. Soc. Sect. B. **70** (1957), 577–589.

It is shown that the Fowler-Darwin definitions of the temperature and chemical potential can be extended in a natural way to cover non-equilibrium but steady assemblies. The concept of equilibrium is replaced by the more general one of 'compatibility', and it becomes possible to show that the proposed statistical definitions of quantities like temperature reduce to suitable averages over the assembly of the locally measured quantities. The usual restrictions to small departures from equilibrium are unnecessary. A simple problem (flow of gas in a tube) is discussed from the new point of view.

Author's summary.

Siegel, Armand. Stochastic basis of the Eulerian variational principle in linear dissipative processes. Phys. Rev. (2) **106** (1957), 609–615.

The ensemble of functions $\alpha(t)$ satisfying Langevin's equation $\dot\alpha(t)+\gamma\alpha(t)=\varepsilon(t)$, with random $\varepsilon(t)$, is studied. In a previous article [Phys. Rev. (2) **102** (1956), 953–959; MR **17**, 1168] the author showed that the sub-ensemble belonging to a fixed value of $\alpha(0)$ clusters around the average behavior, determined by the phenomenological equation $\dot\alpha=-\gamma\alpha$. It is here shown that the subensemble belonging to fixed values of $\alpha(t_1)$ and $\alpha(t_2)$ clusters for $t_1<t<t_2$ around a smooth average behavior determined by $\ddot\alpha=\gamma^2\alpha$. This justifies the variation principle for linear dissipative processes as used by Onsager and Machlup [ibid. **91** (1953), 1505–1512; MR **15**, 273]. It is shown, however, that the number of 'gates' used to describe the path $\alpha(t)$ cannot be increased indefinitely, but must be small compared to the entropy fluctuation (in units k).

N. G. van Kampen (Utrecht).

See also: Hammersley, p. 71; Lifšic, p. 85; Krzywoblocki, p. 88; Bass and Tsidil'kovskii, p. 92; Heitler, p. 96; Szépfalusy, p. 100; Jones, p. 102.

Elasticity, Plasticity

Csonka, P. Contribution to the elastic theory of isotropic bodies. Acta Tech. Acad. Sci. Hungar. **17** (1957), 355–359. (German, French and Russian summaries)

Suppose that the normal components of one symmetric second order Cartesian tensor σ are functions of the normal components of another such tensor ε and that each shear component of σ is a function of the corresponding shear component of ε. Suppose further that σ is an isotropic function of ε. The author shows that, in three dimensions, these conditions hold if and only if $\sigma=2G\varepsilon+f1$, where G is a constant, f is a scalar function of trace ε and 1 is the unit tensor. *J. L. Ericksen* (Washington, D.C.).

Koppe, Eberhard. Methoden der nichtlinearen Elastizitätstheorie mit Anwendung auf die dünne Platte endlicher Durchbiegung. Z. Angew. Math. Mech. **36** (1956), 455–462. (English, French and Russian summaries)

After an exposition of well-known material on finite elastic strain, the author formulates a theory of thin plates based on special assumptions concerning the Piola-Kirchhoff stress tensor (sometimes called the tensor of "pseudo-stress"). Certain non-linear terms result from the fact that this tensor is not symmetric.

C. Truesdell (Bloomington, Ind.).

Deverall, L. I. Solution of some problems in bending of thin clamped plates by means of the method of Muskhelishvili. J. Appl. Mech. **24** (1957), 295–298.

The area of the plate is mapped conformally onto the unit circle $|\zeta|=1$ approximately by polynomials in ζ. The complex potentials are also approximated by polynomials in ζ and equations for the coefficients are determined by the method of Muskhelishvili. Numerical results for the maximum deflection of clamped uniformly loaded square, rectangular and equilateral triangular plates are given and compared with those found previously by other writers. *A. E. Green* (Newcastle-upon-Tyne).

Horvay, Gabriel. Biharmonic eigenvalue problem of the semi-infinite strip. Quart. Appl. Math. **15** (1957), 65–81.

Paper is concerned with the determination of stress and deformation produced by arbitrary self-equilibrating normal and shear tractions acting on the edge $x=0$ of a semi-infinite elastic strip ($0\leq x<\infty$, $-1\leq y\leq 1$) which is free along the edges $y=\pm1$. Problem is solved in two steps. Author first considers the parallel edges $y=\pm1$ of the strip to be free, and along the edge $x=0$, (a) the shear displacement is given, the normal stress is zero, (b) the normal displacement is given, the shear stress is zero. These two problems are solved by extending the strip to

region $-\infty<x\leq 0$, $-1\leq y\leq 1$, and finding tractions at $y=\pm 1$ ($x<0$) and at $x=-\infty$ so that the normal and shear stresses respectively vanish at $x=0$, while the edge values of the displacements are orthogonal polynomials in y. The second phase of the solution consists in recombining stress functions obtained in the first phase so that along the edge $x=0$, (a) the shear stress is given and normal stress is zero, (b) the normal stress is given and shear stress is zero; thereby obtaining two complete orthonormal sets of transcendental functions in y into which the given boundary stresses may be expanded.

A. E. Green (Newcastle-upon-Tyne).

★ **Eringen, A. Cemal. The solution of a class of mixed-mixed boundary value problems in plane elasticity.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 257–265. American Society of Mechanical Engineers, New York, 1955. $9.00.

The paper contains the solution to a novel boundary-value problem of an orthotropic elastic rectangular plate in the state of plane strain. The boundary conditions, which are of a mixed type, consist of normal and tangential displacements on two parallel boundaries, and normal stress and tangential displacement on the other two.

The solution is a very interesting example of the use finite Fourier transforms. *P. M. Naghdi.*

Mossakowski, Jerzy. The Michell problem for anisotropic semi-infinite plate. Arch. Mech. Stos. **8** (1956), 539–548.

Author solves problem of an isolated force in a semi-infinite anisotropic plate bounded by a free straight edge. Results are specialized to problem of isolated force at the edge, either normal or perpendicular to the boundary, and are illustrated numerically. Solution of corresponding problem for clamped edge is also indicated. (Paper contains references to some earlier work for isotropic and orthotropic plates. Author's first problem is solved in "Theoretical elasticity" [Oxford, 1954; MR **16**, 306] by A. E. Green and W. Zerna, where additional references can be found.) *A. E. Green* (Newcastle-upon-Tyne).

Conway, Harry Donald. The indentation of an orthotropic half plane. Z. Angew. Math. Phys. **6** (1955), 402–405.

Daraktschieff, N. Untersuchung des räumlichen achsensymmetrischen Spannungszustandes eines Hohlkugels. Univ. d'Etat Varna Fac. Tech. Méc. Annuaire **4** (1948–1949), 209–250. (Bulgarian. German summary)

In this paper the stresses on the surface of a spherical cavity in an infinite elastic medium are investigated in two cases: 1) if the given stresses at infinity are in all directions constant, equal and tensile, while the surface of the cavity itself is free from load; and 2) if the stresses at infinity are given as follows: $\sigma_x=\sigma_z=-C$ (compressive), $\sigma_y=2C$ (tensile) but the cavity surface is again free from load.

The author finds by simple means suitable space potential functions which determine the axial-symmetric distributions of the stresses on the surface of the cavity.

T. P. Anđelić (Belgrade).

Rongved, L. Dislocation over a bounded plane area in an infinite solid. J. Appl. Mech. **24** (1957), 252–254.

Consider a linear, homogeneous and isotropic, elastic medium which occupies the entire space and possesses a prescribed discontinuity in its displacement field across a bounded plane region R. The author determines the associated Papkovich stress functions with the aid of a potential-theoretic scheme employed previously by Mindlin [Proc. 1st Midwestern Confer. Solid Mech., 1953, Univ. of Illinois, 1954, pp. 56–59; MR **15**, 842]. The representations reached are in terms of the Newtonian potential of a plane mass distribution over R, whose density is proportional to the given displacement discontinuity. Closed formulas are given corresponding to the case in which R is a rectangle, and the dislocation is confined to a uniform jump in the displacement component normal to R. {Reviewer's comment: If R is a circle, and under analogous circumstances, closed representations in terms of elliptic integrals are equally immediate.} No mention is made of the assumptions regarding the behavior of the displacements at infinity, needed to justify the analysis contained in this paper. *E. Sternberg.*

Mohan, R. Some simple problems of flexure. I. Z. Angew. Math. Mech. **36** (1956), 427–432. (English, French and Russian summaries)

St. Venant's flexure problem for an isotropic beam symmetrically loaded is solved for (i) a horse-shoe shaped section bounded by confocal ellipses and part of their common minor axis, (ii) a segment of a parabola bounded by a chord perpendicular to the axis.

L. M. Milne-Thomson (Providence, R.I.).

Koppe, E. Zur nichtlinearen Torsion eines Kreiszylinders. Ing.-Arch. **25** (1957), 1–9.

A tensor generalization of Hencky's logarithmic definition of strain is given. This definition is particularly appropriate for incompressible materials and for such materials the stress strain law (discarding terms of order higher than the second in the strains) takes the form

$$\tau_\epsilon^\beta=p\delta_\epsilon^\beta+\Phi h_\epsilon^\beta+\Psi h_\alpha^\beta h_\epsilon^\alpha,$$

where h_ϵ^β is the strain, Φ, Ψ elastic constants and p the arbitrary hydrostatic pressure. The torsion of a cicular cylinder is treated in some detail and relevent experimental work with rubber described. *R. C. T. Smith.*

Ellington, J. P. On obtaining the shear stress-strain relationship from a hollow specimen in torsion. J. Roy. Aero. Soc. **60** (1956), 806–808.

Čobanyan, K. S. On bending of a composite rod. Akad. Nauk Armyan. SSR. Dokl. **23** (1956), 103–110. (Russian. Armenian summary)

The differential equation and boundary condition (for both internal and external boundaries) for the stress function of bending are derived for a composite prismatic rod, each portion of the rod having the same Poisson's ratio. *R. C. T. Smith* (Armidale).

Seames, A. E.; and Conway, H. D. A numerical procedure for calculating the large deflections of straight and curved beams. J. Appl. Mech. **24** (1957), 289–294.

The authors present a numerical method for solving the Euler-Bernoulli equation, using the full non-linear expression for the radius of curvature of the deformed beam. The method involves replacing the elastic axis by a number of circular arcs, tangent to each other at the points of intersection. A set of simultaneous equations is obtained for each arc, and these can be systematically

solved. The method is illustrated by several examples of straight and curved cantilever beams and of rings subjected to concentrated or distributed loads.

W. E. Boyce (Troy, N.Y.).

Blaise, P. Le calcul des poutraisons dans l'espace et le plan; théorie des réseaux élastiques. Ann. Ponts Chaussées **126** (1956), 445–462.

Les systèmes de poutraisons constitués d'éléments filiformes, solidaires en leurs extrémités et soumis à des systèmes de forces ne causant que des déformations petites, présentent avec les réseaux électriques auxquels s'attache le nom de Kirchhoff des analogies topologiques complètes tandis que les relations entre tensions et déformations, bien que d'un caractère différent des relations entre différences de potentiel et courants, sont linéaires comme elles.

Une similitude des méthodes de calcul doit s'en déduire. C'est ce que nous montrons ci-après en transposant la méthode de détermination de l'état électrique d'un réseau dit de Kirchhoff au calcul de l'équilibre élastique d'un système de poutraisons que nous appellerons „réseau élastique".

Résumé de l'auteur.

Jasiński, F. Investigation of the rigidity of compressed bars. Arch. Mech. Stos. **8** (1956), 319–390. (Polish. Russian and English summaries)

A reprint of the Polish version of the book published simultaneously in Polish and Russian in 1895.

Wierzbicki, Witold. The contribution of Feliks Jasiński to world science. Arch. Mech. Stos. **8** (1956), 293–317. (Polish. Russian and English summaries)

Chiefly a discussion of the book listed above.

Jasiński, Stefan. Feliks Jasiński (1856-1899), esquisse biographique. Arch. Mech. Stos. **8** (1956), 259–291 (1 plate). (Polish. Russian and French summaries)

This biographical essay, published as a preface to the book listed second above, is accompanied by a photograph of Jasiński and by ten sketches of his most famous structures, for example, of the Gatczino railroad station.

★**Fletcher, H. J.; and Thorne, C. J. Bending of thin rectangular plates.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 389–406. American Society of Mechanical Engineers, New York, 1955. $9.00.

The paper contains a systematic treatment of a large class of problems (within the scope of the classical theory of bending of plates) which have previously been treated piecemeal. A Fourier series solution is obtained for rectangular plates under transverse load with each edge either free, simply supported, or clamped, which leads to an infinite set of equations with an infinite number of unknowns. A discussion of convergence is included.

P. M. Naghdi (Ann Arbor, Mich.).

Buchwald, V. T. A mixed boundary-value problem in the elementary theory of elastic plates. Quart. J. Mech. Appl. Math. **10** (1957), 183–190.

The paper is concerned with problems in the theory of bending of a thin half-plane when the long straight edge is partly clamped and partly simply supported with, or without, the presence of isolated loads. *A. E. Green.*

Woinowsky-Krieger, S. Über die Biegung des orthotropen Plattenstreifens durch Einzellasten. Ing.-Arch. **25** (1957), 90–99.

M. T. Huber [Probleme der Statik technisch wichtiger orthotroper Platten, Warsaw, 1929] gave a simple Fourier series expression for the deflection of an infinite orthotropic strip under a point load. The present author reduces this expression to a more useful closed form. Similar results are given for a semi-infinite strip with the short edge either clamped or simply supported.

R. C. T. Smith (Armidale).

Dmitriev, A. D. Approximate solution of the problem of an infinite plate on an elastic foundation in the case of axial-symmetric load distribution. Trudy Saratov. Avtomobil'nodorož. Inst. **13** (1955), 25–50. (Russian)

The exact solutions of the fourth order partial differential equation for an infinite plate under axi-symmetric load contain Bessel and Henkel functions. Exact solutions were given by S. A. Bernstein in "18th Collection of the Department of Engineering Research of Techno-Scientific Committee NKPS" [Transpečatat', 1929], and by B. G. Korenev in "Circular plates on elastic foundation" [Research in Engineering Designs, 1st ed., Stroizdat, 1948].

Bessel and Henkel functions are not suitable for numerical computations and an alternative easy to compute was desired. V. A. Gastev in "Proceedings of War-Transport Academy" [2nd ed., Transželedorizdat, 1944], presented such an approximate solution. But he changed the original problem of an infinite plate on elastic foundation into a problem of a plate with finite radius fixed elastically on the boundaries.

The author of this paper gives another approximate solution, which is easy to compute, without changing the original problem. He constructs a displacement function satisfying all the boundary conditions. This function contains two parameters which are determined from the energy and equilibrium equations. The deflection curve obtained in this way differs at the most by 3% from the exact one. The first solution assumes a concentrated force and it is not suitable for computing the moments and the shearing force. The author finds then another solution for a load distributed over a circular area of a small radius, from which the moments and the shearing force are easily computed. Comparison with the exact solutions shows only very small deviations. *T. Leser* (Aberdeen, Md.).

Thurston, G. A. Bending and buckling of clamped sandwich plates. J. Aero. Sci. **24** (1957), 407–412.

The strain energy expression for sandwich plates which is due to Hoff [NACA Tech. Note no. **2225** (1950); MR **12**, 561] is used to derive formulas for the deflections and buckling loads of rectangular plates clamped on all four edges. Since an exact solution would lead to an infinite determinant, approximate methods are considered. The usual Ritz procedure results in an upper bound solution. A lower bound solution is obtained using a principle from the calculus of variations due to Courant and Hilbert [Methoden der mathematischen Physik, Bd. 1, 2. Aufl., Springer, Berlin, 1931, p. 353] and the Lagrangian multiplier method developed by Budiansky and Hu [NACA Tech. Note no. **1103** (1946); MR **8**, 118].

Numerical calculations were performed on the I.B.M. 704 computer to obtain the buckling coefficients for the lower bound solution. Curves are obtained for buckling loads which are suitable for use in design. Comparison o

the theory with published test data indicated good agreement between theory and experiment.

In view of the good accuracy obtained with the lower bound solution, the numerically more complex upper bound solution was not programmed. Equations are also obtained which are suitable for obtaining stresses and deflections in clamped sandwich plates under transverse loads. *A. M. Garber* (Philadelphia, Pa.).

Kaliski, Sylwester; and Nowacki, Witold. Some problems of structural analysis of plates with mixed boundary conditions. Arch. Mech. Stos. **8** (1956), 413–448.

In a general statement of the problem of bending of plates with mixed, discontinuous boundary conditions the authors select among the various methods the representation by a system of linear integral equations for a set of unknown functions, which determine the deflection of the plate. These Fredholm equations are of the first kind, when the supports at the edge of the plate are rigid, and of the second kind, when they are elastic. A number of problems are solved, where the kernels of the Fredholm equations are known, i.e. the rectangular plate and the infinite strip with various edge conditions. In the case of the elastic supports the method of approximation by iterated kernels is applied, and numerical values of the estimated errors for the edge reactions and moments are given. *W. Schumann* (Zürich).

Iškova, A. G.; and Tulaĭkov, A. N. Certain problems of bending of plates resting on an elastic half-space. Inžen. Sb. **23** (1956), 47–62. (Russian)

★ **Ôhasi, Yosio. Bending of a thin elliptic plate of an orthotropic material under uniform lateral load.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 163–167. Science Council of Japan, Tokyo, 1952.

Rüdiger, D. Spannungen und Verschiebungen der krummen Flächen mit elliptischem Grundriss. Österreich. Ing.-Arch. **10** (1956), 66–74.

Rüdiger, D. Spannungen und Verschiebungen der krummen Flächen mit schiefem Grundriss. Österreich. Ing.-Arch. **9** (1955), 265–273.

Stippes, M. A note on the simply-supported plate. Quart. Appl. Math. **14** (1956), 90–93.

Szmodits, K. Voiles minces sans poussée latérale construits sur des surfaces elliptiques. Acta Tech. Acad. Sci. Hungar. **13** (1955), 327–335. (Russian, English, and German summaries)

★ **Washizu, Kyuichiro. On the bending of orthogonally aeolotropic plates.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 157–162. Science Council of Japan, Tokyo, 1952.

Woinowsky-Krieger, S. Über die Verwendung von Bipolarkoordinaten zur Lösung einiger Probleme der Plattenbiegung. Ing.-Arch. **24** (1956), 47–52.

Salvadori, Mario G. Boundary displacements in membranes of revolution under symmetrical loads. Ann. Mat. Pura Appl. (4) **39** (1955), 121–126.

Tungl, E. Membranspannungszustand im elliptischen Paraboloid. Österreich. Ing.-Arch. **10** (1956), 308–314.

Layrangues, M. Etude générale de la déformation élastique des voiles minces. Ann. Ponts Chaussées **126** (1956), 39–76.

Kennard, E. H. Approximate energy and equilibrium equations for cylindrical shells. J. Appl. Mech. **23** (1956), 645–646.

Gerard, George. Compressive and torsional buckling of thin-wall cylinders in yield region. NACA Tech. Note no. **3726** (1956), 42 pp.

Clark, R. A.; and Reissner, E. On axially symmetric bending of nearly cylindrical shells of revolution. J. Appl. Mech. **23** (1956), 59–67.

Eringen, A. Cemal. Response of an elastic disk to impact and moving loads. Quart. J. Mech. Appl. Math. **8** (1955), 385–393.

Mindlin, R. D.; Schacknow, A.; and Deresiewicz, H. Flexural vibrations of rectangular plates. J. Appl. Mech. **23** (1956), 430–436.

Cox, Hugh L. Vibration of certain square plates having similar adjacent edges. Quart. J. Mech. Appl. Math. **8** (1955), 454–456.

Yu, Yi-Yuan. Dynamic equations of Donnell's type for cylindrical shells with application to vibration problems. Syracuse University Research Institute, Mechanical Engineering Department, Rep. No. ME390-5610TN1 (1956). iv+22 pp.

Starting from the Flügge equations, the author developes a set of Donnell-type dynamic equations for cylindrical shells. He also developes a set which, unlike the above, includes transverse shear and rotational inertia effects, and shows that the second set reduces to the first.

Ignoring transverse shear and rotational inertia, he then investigates the free vibrations of cylindrical shells with freely supported edges. Clamped and flexibly supported edges are then discussed by Galerkin's method, and it is shown that the method is very suitable for dealing with Donnell-type equations. *H. D. Conway.*

Mettler, E.; und Weidenhammer, F. Der axial pulsierend belastete Stab mit Endmasse. Z. Angew. Math. Mech. **36** (1956), 284–287.

The authors consider the transverse vibrations of a bar which is subject to axial pulsations and carries a terminal mass at the moving end. The differential equations for the longitudinal and transverse motions are nonlinear since the influence of transverse displacement on longitudinal strain is exhibited. One end of the bar is attached to a fixed, simple support whereas three cases are considered for the other end: a) harmonic end motion, b) harmonic end force, c) terminal mass with harmonic end force.

Kirchhoff's assumption, that the longitudinal strain is only time dependent, leads to a non-linear integro-differential equation. The equation is analyzed by a perturbation method for deriving frequencies in the neigh-

borhood of twice the lowest natural bending frequency. The frequency-amplitude curve for case c) is sublinear in that increasing amplitudes are possible only for decreasing driving frequencies. Case b) is linear, whereas case a) is superlinear. Stability calculations are also indicated.

The problem is also analyzed with the assumption of an inextensible center line and the effect of damping is considered. The results compare well with those given by the Kirchhoff assumption.

The authors note in a footnote that these results show that some of their preceding work [Weidenhammer, Ing.-Arch. **20** (1952), 315–330; **24** (1956), 53–68; MR **14**, 1039; **17**, 1254; Mettler, ibid. **23** (1955), 354–364; MR **17**, 431] must be reinterpreted as to the boundary conditions imposed at the driving end. *G. H. Handelman.*

Heidenhain, H. Über den Einfluss einer Endmasse und Endfeder auf die Frequenz-Amplituden-Abhängigkeit längserregter Saitenquerschwingungen. Z. Angew. Math. Mech. **36** (1956), 280–282.

The author considers the longitudinal and transverse motion of an elastic string which is fixed at one end and is subject at the other end to the combined effect of the following: an axially moving terminal mass, a constant axial force, a harmonic axial force, and a linear spring which can move axially. The coupled differential equations for the two motions are non-linear because of the coupling terms. As in the paper reviewed above the Kirchhoff assumption that the longitudinal force is only time dependent is used. Perturbation methods are used to study solutions in which the frequency of the driving force is in the neighborhood of twice the lowest frequency of the natural transverse vibration. The resulting amplitude-frequency curve is sublinear, linear, or superlinear depending on the relative importance of the terminal mass or the terminal spring. The results agree closely whit experiment. *G. H. Handelman* (Troy, N.Y.).

Morrow, Charles T. Shock spectrum as a criterion of severity of shock impulses. J. Acoust. Soc. Amer. **29** (1957), 596–602.

Haskins, J. F.; and Walsh, J. L. Vibrations of ferroelectric cylindrical shells with transverse isotropy. I. Radially polarized case. J. Acoust. Soc. Amer. **29** (1957), 729–734.

Expressions for the coupled mechanical vibrations and electrical admittance of ferroelectric tubes having transverse isotropy are derived and the results supported with experimental data. (From the author's summary.)

H. Levine (Stanford, Calif.).

Naleszkiewicz, Jarosław. Energy levels in dynamics of elastic systems. Arch. Mech. Stos. **8** (1956), 471–506.

In part one the author discusses a class of elastic structures having a number of dynamically stable equilibrium configurations. If the structure is perturbed, the configuration selected by the structure depends on the energy of the initial displacement and the damping law. The motion of a certain spring pendulum is treated at some length to illustrate the various possibilities.

Part two concerns the coupled bending and torsional vibrations of a beam loaded by an axial force and a moment. Two cases are distinguished, on the basis of whether the applied moment has a twisting component or is purely a bending moment. Both problems are accompanied by numerous computed results.

W. E. Boyce (Troy, N.Y.).

Handelman, George; and Tu, Yih-O. On the antisymmetric vibrations of a beam carrying a distributed added mass. J. Appl. Mech. **24** (1957), 312–313.

This is another in a series of papers by the first author and associates [Kornhauser and Mintzer, J. Acoust. Soc. Amer. **25** (1953), 903–906; Cohen and Handelman, ibid. **27** (1955), 177; J. Franklin Inst. **261** (1956), 319–329; MR **17**, 916] on the effect of adding a distributed mass to a vibrating system. In the present case the mass is assumed to be added over a central zone of the beam. A minimum principle is used to obtain an approximate formula for the frequency. Computed results show that, as the length of the interval under the mass is increased, the ultimate effect will be to increase the frequency by effectively shortening the beam; however, initially the frequency may be either raised or lowered, depending on the density of the added mass. *W. E. Boyce* (Troy, N.Y.).

Ralston, Anthony. On the problem of buckling of a hyperbolic paraboloidal shell loaded by its own weight. J. Math. Phys. **35** (1956), 53–59.

Klein, Bertram. Buckling of simply-supported rhombic plates under externally applied shear. J. Roy. Aero. Soc. **61** (1957), 357–358.

An approximate solution by the method of collocation is given. *R. C. T. Smith* (Armidale).

Heinrich, G. Zur Stabilität der Strickleiter. Österreich. Ing.-Arch. **10** (1956), 175–189.

Es wird das Gleichgewicht und die Stabilität einer „kontuinierlichen Strickleiter" untersucht, die an einem Ende festgehalten ist und am anderen, freien Ende durch zwei konservative Kräfte belastet ist. Die erste Variation eines bestimmten Integrals liefert die möglichen Gleichgewichtsfälle, die zweite Variation liefert ein Stabilitätskriterium. Dieses wird für den Fall der schraubensymmetrischen Lösung angegeben.

Zusammenfassung des Autors.

Synge, J. L. Elastic waves in anisotropic media. J. Math. Phys. **35** (1957), 323–334.

The general anisotropic homogeneous elastic solid is described by 21 elastic constants (which in the case of isotropy reduce to 2). The author considers infinitesimal plane waves travelling through such a medium. Equations are given for the wave surface, the velocity surface, and the slowness (=reciprocal wave) surface. Next the author studies forced waves in a layer of constant thickness which are excited by stress waves travelling over one of the faces. Conditions are found for resonance and for the existence of free vibrations: If the thickness of the layer and the period are given, there is a discrete set of free vibrations (possibly null), each with a definite surface velocity vector. As the thickness of the layer tends to infinity, some of the free modes may tend to surface (Rayleigh) waves. The familiar Rayleigh waves in an isotropic medium form a highly degenerate limiting case, the locus of possible velocities being a circle and not a number of isolated points. In a medium with transverse isotropy (5 elastic constants) the slowness surface breaks up into a spheroid and a quartic surface. *F. Ursell.*

Manuhov, A. V. On the approximation of thin layers by degenerate models. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 1400–1410. (Russian)

This paper studies the propagation of seismic waves in

a liquid half-space $z>0$ containing at the depth $z=H$ an infinite elastic layer of thickness h under the assumption that the wave-length λ is much greater than h. Replacing in the first approximation the elastic layer by a membrane the author, obtains very definite conclusions which — as he states — were completely confirmed by experiments.
E. Kogbetliantz (New York, N.Y.).

Hodge, P. G., Jr. Piecewise linear isotropic plasticity applied to a circular cylindrical shell with symmetrical radial loading. J. Franklin Inst. **263** (1957), 13–33.

The paper is concerned with the work-hardening behavior of a circular cylindrical shell with radial loading. The yield condition is assumed to be represented by a square in the relevant stress space. The rates of stress and strain satisfy piecewise linear relationships for various plastic states represented by the yield curve so that a direct equation may be obtained for stresses as functions of strains. It is then shown that under certain restrictions on the loading, the two classical minimum principles of minimum potential and complementary energy are valid. These minimum principles are applied to an example and compared with the exact solution. *E. T. Onat.*

Boyce, William E. The bending of a work-hardening circular plate by a uniform transverse load. Quart. Appl. Math. **14** (1956), 277–288.

The paper contains an analysis of the bending moments and deflection of a work-hardening circular plate under the action of a uniformly distributed transverse load. The mechanical behavior of a plate element is discussed in terms of the yield curve and its strain-history dependence in a Cartesian system with the radial and circumferential bending moments as coordinates. The virgin yield curve is approximated by an octagon. It is assumed that the yield curve at a given stage of loading process is obtained by a rigid body translation of the virgin yield octagon. The translatory motion of the yield curve is related to the changes of curvature in the radial and circumferential direction in the manner described by Prager [J. Appl. Mech. **23** (1956), 493–496; MR **18**, 688] so that the unsound features of total stress-strain laws are avoided while much of their mathematical simplicity is retained. The bending of a uniformly loaded, simply supported circular plate is then studied in detail using the work-hardening behavior described above. *E. T. Onat.*

Tekinalp, Bekir. Elastic-plastic bending of a built-in circular plate under a uniformly distributed load. J. Mech. Phys. Solids **5** (1957), 135–142.

Title problem is analysed for the case of an incompressible material obeying Tresca's yield condition and the associated flow rule. The work is simplified by assuming that any plate element is either entirely elastic or entirely plastic. As in the reviewer's analysis of the simply supported plate with a central concentrated load [Proc. 2nd U.S. Nat. Congress Appl. Mech., Univ. of Michigan, 1954, Amer. Soc. Mech. Engrs., New York, 1955, pp. 521–526], no account is taken of the development of membrane stress due to changes in shape. Recent theoretical and experimental work [J. Appl. Mech. **23** (1956), 49–55] indicates that shape changes become significant when the deflection reaches a small fraction of the thickness of the plate. Thus deflection studies in which shape changes are neglected are unlikely to produce results having physical significance.
R. M. Haythornthwaite (Providence, R.I.).

Freiberger, W.; and Prager, W. Plastic twisting of thick-walled circular ring sectors. J. Appl. Mech. **23** (1956), 461–463.

The graphical method [J. Mech. Phys. Solids **3** (1955), 169–175; MR **16**, 1073] of Wang and Prager [based on Freiberger's characteristic equation, Commonwealth of Australia, Aero. Res. Lab., Rep. **ARL SM 213** (1953); Quart. Appl. Math. **14** (1956), 259–265; MR **18**, 436] for determining the fully plastic stress distribution and velocities in a twisted circular ring sector of solid section is extended to circular ring sectors of hollow cross-section. The solution illustrates once more the important role of discontinuities in the theory of perfectly plastic solids [Prager, Proc. 2nd U.S. Nat. Congress Appl. Mech., Ann Arbor, 1954, Amer. Soc. Mech. Engrs., New York, 1955, pp. 21–32; MR **18**, 82]. *P. M. Naghdi.*

Arčašnikov, V. P. On a problem of the theory of limiting equilibrium. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. **1956**, no. 5(9), 109–115. (Russian)

Sanoyan, Z. G. On the theory of motion of suspended drifts. Akad. Nauk Armyan. SSR. Dokl. **23** (1956), 11–15. (Russian. Armenian summary)

Nowacki, Witold. The stresses in a thin plate due to a nucleus of thermoelastic strain. Arch. Mech. Stos. **9** (1957), 89–106. (Polish and Russian summaries)

The stresses in a thin plate due to a nucleus of thermoelastic strain is studied by means of Green's function associated to the problem. The author solves the cases of an infinite strip, a semi-infinite strip and a semi-infinite plate by using trigonometric series and Fourier integrals.
W. Schumann (Zürich).

See also: Yu, p. 28; Shuleshko, p. 41; Sabroff and Higgins, p. 64; Thomson, p. 77; Godfrey, p. 77; Lliboutry, p. 104; Chopra, p. 104.

Structure of Matter

★ **Lonsdale, Kathleen. Thermal vibrations of atoms and molecules in crystals.** International conference on current problems in crystal physics. pp. 244–253. Massachusetts Institute of Technology, Cambridge, Mass., July 1–5, 1957.

The determination of the effect of temperature on the vibrations of atoms and molecular groups in crystals is reviewed. In the case of simple crystals the effect of zero-point vibrations on the X-ray structure factors can be evaluated, but this has not been possible as yet for molecular crystals. Experimental measurements on diffraction patterns must be improved in accuracy before reliable analyses can be made of the variation of vibrational motions with temperature. The author gives a brief review of the types of measurement of greatest significance.
E. L. Hill (Minneapolis, Minn.).

Pyle, I. C. The second-order effect of free electrons on lattice conduction. Proc. Cambridge Philos. Soc. **53** (1957), 508–513.

Second-order perturbation theory is applied to test whether processes involving the virtual absorption and re-emission of phonons by free electrons, or the converse processes, are of importance in a semiconductor or metal. The conclusion is that they are of negligible numerical

significance. The work is an extension of that of J. M. Ziman [Phil. Mag. (8) 1 (1956), 191–198]. *E. L. Hill.*

Lifšic, M. Some problems of the dynamic theory of non-ideal crystal lattices. Nuovo Cimento (10) 3 (1956), supplemento, 716–734.

A theory for calculating the effect of localized perturbations (such as impurity atoms) on crystal lattice waves (phonons) is presented along lines developed by the author in previous papers, and later published independently by M. Lax [Phys. Rev. (2) **94** (1954), 1391–1392], Koster and Slater [Koster, ibid. **95** (1954), 1436–1443; Koster and Slater, ibid. **94** (1954), 1392; **95** (1954), 1167–1176]. He then discusses the extension of his method to disordered crystals, and gives a formula developing the free energy, and at least formally the spectrum, in powers of the impurity concentration. *P. W. Anderson.*

Parrott, J. E. Some contributions to the theory of electrical conductivity, thermal conductivity and thermoelectric power in semiconductors. Proc. Phys. Soc. Sect. B. **70** (1957), 590–607.

The theory of transport phenomena in semiconductors is extended to include the effects of 'phonon drag' on electrons and 'electron drag' on phonons.

From the author's summary.

Bertaut, E. F.; et Dulac, J. Tables de linéarisation des produits et puissances des facteurs de structure. Acta Cryst. 9 (1956), 322–323.

The trigonometric parts $\xi(h_j)$ of structure factors form the base of an algebra in the sense that they satisfy relations of the type

$$\xi(h_j)\xi(h_k)=\sum_{l=1}^{n} g_{jkl}\xi(h_l).$$

In statistical methods of sign determination it is necessary to find the mean values of products of powers of structure factors, and hence the constant term (coefficient of $\xi(0)$) in a linearization relation

$$\xi^a(h_1)\xi^b(h_2)\cdots\xi^k(h_m)=\sum_s a_s\xi(H_s).$$

A duplicated table (pp. 93) has been prepared containing explicitly the simpler linearization relations for the 230 space groups (one or two factors, power at most 4, usually 1 or 2) and the plane groups. *A. J. C. Wilson.*

Cochran, W. Scattering of X-rays by defect structures. Acta Cryst. 9 (1956), 259–262.

A spherical perfect crystal C of radius $\mathscr{R}$ is modified by the introduction of defects Δ, in general an assemblage of negative atoms at the points from which atoms have been displaced, and positive atoms at the points to which atoms have been displaced. The Fourier transform of [the electron density in] the defective crystal is

$$T_{C+\Delta}(\mathbf{S})=T_C(\mathbf{S})+T_\Delta(\mathbf{S}),$$

where $\mathbf{S}$ is a vector in reciprocal space and

$$T_\Delta(\mathbf{S})=\sum_L\sum_m[f'_{m,L}\exp\{2\pi i(\mathbf{R}_L'+\mathbf{r}'_{m,L})\cdot\mathbf{S}\}+ \\ +f_m\exp\{2\pi i(\mathbf{R}_L+\mathbf{r}_m)\cdot\mathbf{S}\},$$

where $\mathbf{R}_L$ is the position vector of a unit cell, $\mathbf{r}_{m,L}$ is the position vector of the mth atom within it, and primes distinguish atoms of the defective crystal. If $\mathbf{H}$ is a vector of the reciprocal lattice, $F(\mathbf{H})$ the structure factor of C, and $F_M(\mathbf{H})$ the mean structure factor of $C+\Delta$ [loosely called 'structure factor of the average unit cell' by the author], then

$$T_\Delta(\mathbf{H})=N\{F_M(\mathbf{H})-F(\mathbf{H})\},$$

where N is the total number of cells. The transform of a crystal of radius $\mathscr{R}$ composed of identical cells of mean structure factor is

$$T_M(\mathbf{S})=T_L(\mathbf{S})F_M(\mathbf{H}),$$

where T_L is the transform of the lattice only, so that

$$T_{C+\Delta}(\mathbf{S})=T_M(\mathbf{S})+\{T_\Delta(\mathbf{S})-N^{-1}T_L(\mathbf{S})T_\Delta(\mathbf{H})\}.$$

The two terms on the right are never large simultaneously; the first is appreciable only within about $\mathscr{R}^{-1}$ of the points of the reciprocal lattice, where the second vanishes. To a close approximation, therefore, the X-ray diffraction maximum is the sum of a sharp reflexion of intensity

$$J_1(\mathbf{S})=|T_M(\mathbf{S})|^2$$

and a diffuse background of intensity

$$J_2(\mathbf{S})=|T_\Delta(\mathbf{S})-N^{-1}T_L(\mathbf{S})T_\Delta(\mathbf{H})|^2\sim|T_\Delta(\mathbf{S})|^2,$$

since the second term is appreciable only within about $\mathscr{R}^{-1}$ of $\mathbf{S}=\mathbf{H}$. This result is equivalent to that of W. H. Zachariasen [Theory of X-ray diffraction in crystals, Wiley, New York, 1945], but is in a form more suitable for detailed calculation. Under the assumption of linear additivity of not-too-large displacement it is shown that the effect of all the defects making up Δ is the sum of their effects considered separately.

The problem of the random replacement of atoms by others of a different size and scattering power, which has applications to disordered alloys, is treated in detail, with results qualitatively and quantitatively in close agreement with those of K. Huang (Proc. Roy. Soc. A, **190**, 102, 1947); small differences in detail are ascribed to differences in the approximations made. Applications to neutron-irradiated crystals and to Frenkel defects are sketched. *A. J. C. Wilson* (Cardiff).

See also: Shabanskii, p. 78; Bass and Tsidil'kovskii, p. 92; Gell-Mann and Brueckner, p. 98; Gell-Mann, p. 98; Sawada, p. 98; Samoilovich and Kond ratenko, p. 101; Jones, p. 102.

Fluid Mechanics, Acoustics

Janne d'Othée, Henry. Les lois de conservation de la physique, leurs combinaisons et leur éventuelle interdépendance. Bull. Soc. Roy. Sci. Liège **25** (1956), 483–513.

The author considers conservation principles in continuous media, and especially in the case of viscid, heat conducting liquids (with constant coefficients of viscosity). The consequences of conservation principles in the case of discontinuities (shock waves) are also discussed. The paper is connected with ideas given by R. Courant and K. O. Friedrichs, [Supersonic flow and shock waves, Interscience, New York, 1948, ch. III; MR **10**, 637].

J. Plebański (Warsaw).

Landweber, L.; and Yih, C. S. Forces, moments, and added masses for Rankine bodies. J. Fluid Mech. 1 (1956), 319–336.

The dynamical theory of the motion of a body through

an inviscid and incompressible fluid has yielded three relations; a first due to Kirchhoff which expresses the force acting on the body in terms of added masses; a second initiated by Taylor which expresses added masses in terms of singularities within the body; and a third due to Lagally which expresses the force in terms of these singularities. In this paper the authors are concerned with generalizations of the Taylor and Lagally theorems to include unsteady flow and arbitrary motions of the body. Taylor's theorem is derived when other boundaries are present. For the added mass coefficients due to pure rotation, for which no relations were known, the authors claim to have shown that these relations do not exist in general. This non-existence proof, in the two-dimensional case, appears, however, to the reviewer to depend on definition, for the integral on which the proof is based can in fact be treated by the residue theorem if one observes that on the contour the conjugate complex of z is a function of z. The derivation of the Lagally theorem leads to complete expression for the force on elongated bodies in terms of singularities. *L. M. Milne-Thomson.*

Haskind, M. D. Three-dimensional flow about thin bodies. Prikl. Mat. Meh. **20** (1956), 203–210. (Russian)

Potential theory methods are used to formulate a singular integro-differential equation for a in a cambered lifting line in a plane in a three-dimensional flow of an unlimited weightless fluid. Using methods of linear wave theory, equation is generalized to case of slender body motion in fluid with weight. Plane in which cambered lifting line being considered lies is perpendicular to flow plane. Dorodnitsyn [Prikl. Mat. Meh. **8** (1944), 33–64; MR **8**, 110] considered another case of a cambered lifting line in the flow plane.

Use of potential-theoretic methods permits an equation to be obtained for the cambered lifting line; hence, assumptions are explained to which the derivation of this equation is related. Solutions are given for this equation for a lifting line in the shape of half a ring for small and large Froude numbers. Also given is an approximate solution taking Froude number into account for a vertical rectilinear lifting line completely submerged in a heavy fluid.

Using these methods, Küssner [Luftfahrtforschung **17** (1940), 370–378; MR **2**, 330] established the generalized Prandtl integro-differential equation for a rectilinear lifting line in the unsteady flow case. (From author's summary.) *M. D. Friedman* (Newtonville, Mass.).

Gibbings, J. C.; and Dixon, J. R. Two-dimensional contracting duct flow. Quart. J. Mech. Appl. Math. **10** (1957), 24–41.

This paper deals with the incompressible potential flow through two-dimensional contracting channels of finite length. These channel flows are obtained by first specifying flow patterns in the logarithmic hodograph plane, $\Omega = \log g - i\theta$. It is shown that certain of these patterns can result in infinite values, at points on the channel wall, of both the velocity gradient and wall curvature. A method of avoiding these undesirable features is given which alters the contraction boundary so as the replace it by wall portions along which the velocity gradient is constant. A particular numerical example is worked out by relaxation, and a method is given whereby the value of the gradient can be more readily controlled.

L. M. Milne-Thomson (Providence, R.I.).

Wu, T. Yao-tsu. A free streamline theory for two-dimensional fully cavitated hydrofoils. J. Math. Phys. **35** (1956), 236–265.

The author discusses in considerable detail the free-streamline theory for two-dimensional obstacles (hydrofoils), under the assumption of non-zero cavitation number. A major part of the interest in the work is due to the fact that it deals effectively with obstacles at varying angles of attack to the uniform stream. For reasons of mathematical simplicity, the author approximates the wake downstream of the obstacle by means of the Roshko model [NACA Tech. Note no. **3168** (1954); see also Eppler, J. Rational Mech. Anal. **3** (1954), 591–644; MR **16**, 188], but he remarks that other models give essentially the same results so far as lift and drag are concerned. The mathematical considerations of the paper depend on the conformal mapping of the hodograph plane on the plane of the complex potential; being a generalization of Levi-Civita's classical approach, the flow problem is finally reduced to a non-linear boundary value problem for an analytic function in the unit circle. This problem is treated explicitly in two cases, that of a circular arc and a flat plate; from the final result the effects of various quantities such as the cavitation number, camber of the profile and attack angle, are discussed in detail.

The paper also contains an interesting discussion of various flow models which have been proposed to represent the wake downstream from an obstacle in a cavitating flow. *J. B. Serrin* (Minneapolis, Minn.).

Weinig, F. S. A new approach to the theory of thin, slightly cambered profiles. J. Appl. Mech. **24** (1957), 177–182.

Let $\chi(z)$ be the complex potential when a thin slightly cambered profile $y=P_n(x)$, where $P_n(x)$ is a polynomial of degree n, disturbs a unit flow in the x-direction. The linearized boundary condition is then $dy/dx=-\mathrm{Im}\{\chi'(z)\}$ so that $d^ny/dx^n=-\mathrm{Im}\{\chi^{(n)}(z)\}$ is constant. It is easy to establish such functions of the complex variable in the case of single as well as cascade profiles. Integration then yields the required results. Applications are made to a single circular-arc profile, a single S-profile, and a circular arc profile cascade. *L. M. Milne-Thomson.*

Helmbold, H. B. Theory of the finite-span blowing wing. J. Aero. Sci. **24** (1957), 339–344, 370.

The first part of the paper reviews briefly the American work on the two-dimensional jet flap. In this case the direction of the jet, at infinity, is parallel to the free stream direction. Next the author shows that, for an aerofoil with a finite span, the jet is not parallel to the free stream direction at infinity. The consequences of this result are then investigated and, amongst other things, it is shown that, in the three-dimensional case, there is less useful jet thrust and an induced drag arises.

G. N. Lance (Southampton).

Woods, L. C. Aerodynamic forces on an oscillating aerofoil fitted with a spoiler. Proc. Roy. Soc. London. Ser. A. **239** (1957), 328–337.

The results of a previous study of unsteady flow past obstacles with unsteady wakes [same Proc. **229** (1955), 152–180; MR **16**, 1061] are drawn upon to construct an approximate theory of oscillating airfoils with streamline separation. The separation points are presumed to be fixed, one at the trailing edge and the other at an arbitrary point on the upper surface. The free streamlines are

approximated by their steady-flow configurations. When the upper-surface separation point occurs at the trailing edges, the results agree with classical theories. Numerical results for the damping derivatives are tabulated and plotted here for a range of reduced frequencies and six values of a parameter specifying the separation-point location on the upper surface. It is pointed out that there is generally less damping than in the classical case.

The investigation involved numerical tabulation of the function

$$\int_0^\infty \exp[-i\omega(\cosh x-\alpha \cosh \tfrac{1}{2}x)]\cosh \tfrac{1}{2}nx\, dx$$

for $n=0$, 1, and 2, within the range $-4\leq\alpha\leq4$, $0\leq\omega\leq1$. This was carried out at the National Physical Laboratory. These tables, however, are not published here.

W. R. Sears (Ithaca, N.Y.).

Eppler, R. Direkte Berechnung von Tragflügelprofilen aus der Druckverteilung. Ing.-Arch. **25** (1957), 32–57.

This method is basically similar to Lighthill's [Aero. Res. Council, Rep. and Memo. no. **2112** (1945)], making use of the properties of the logarithm of the complex velocity. The author claims that his technique makes it possible to specify any velocity distribution, without restrictions other than the closure condition, which is applied in the calculation. A method given previously by the same author [Ing.-Arch. **23** (1955), 436–452] was approximate in that the prescribed velocity distribution was achieved only up to a factor $\cos\theta$, where θ is the local profile slope. The present method eliminates this approximation, and yet the numerical labor required by it is no greater. No iteration is required. The method is outlined particularly for three different classes of "laminar-boundary-layer profiles", i.e. for velocity distributions of three different classes, and for the region near a trailing edge. A numerical example falling into the most complicated of these classes is worked out in detail. *W. R. Sears.*

Mangler, K. W.; and Spencer, B. F. R. Some remarks on Multhopp's lifting-surface theory. Aero. Res. Council, Rep. and Memo. no. **2926** (1952), 9 pp. (1956).

This note refers to Multhopp's method [same Rep. and Memo. no. **2884** (1950); MR **17**, 313] for calculating the lift distribution of a wing of finite span. Multhopp assumed the chordwise distribution function and used trigonometric series to calculate the downwash. Actually the integration over the span involves a logarithmic singularity, and a correction term for the trigonometric series was included. Here an alternative method is proposed, which retains Multhopp's scheme of calculation but alters some of the numerical values. The new method is believed to rest on a sounder mathematical basis, but the actual difference is numerical results, as illustrated by examples, is small. *W. R. Sears* (Ithaca, N.Y.).

Drischler, Joseph A.; and Diederich, Franklin W. Lift and moment responses to penetration of sharp-edged traveling gusts, with application to penetration of weak blast waves. NACA Tech. Note no. **3956** (1957), 85 pp.

Weber, J.; and Hawk, A. C. Theoretical load distributions on fin-body-tailplane arrangements in a side-wind. Aero. Res. Council, Rep. and Memo. no. **2992** (1954), 58 pp. (1957).

★**Stoker, J. J. Some recent progress in the theory of surface waves in water.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 251–263. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

This paper discusses a number of problems in the theory of surface waves which has aroused interest in the past few years. The problems considered are (a) diffraction of a plane wave by a semi-infinite barrier, (b) motion of ships as floating rigid bodies, (c) the solitary wave, and (d) river regulation and flood prediction problems.

A. E. Heins (Pittsburgh, Pa.).

Rosenblatt, Murray. A random model of the sea surface generated by a hurricane. J. Math. Mech. **6** (1957), 235–246.

The stated object of this paper is to construct a random model of the sea surface generated by a hurricane. The term "hurricane" must be interpreted very loosely and it would be preferable to use "disturbance" instead. The disturbance is taken be a random process stationary in time and angular dependence; the origin of the coordinate system is at the "storm" center which is fixed. The linearized equations for a constant depth ocean are examined. Instead of specifying a forcing function acting on the ocean surface, the author considers solutions of the linear homogeneous equations which have appropriate singularities and builds these into random potential functions. The character of these representations are investigated for large distances from the disturbance center. *H. Greenspan* (Cambridge, Mass.).

Glauert, M. B. The wall jet. J. Fluid Mech. **1** (1956), 625–643.

The term "wall jet" is used to describe the flow that develops when a jet of fluid in a medium consisting of a similar fluid strikes a surface at right angles and spreads out over it. The appropriate boundary layer equations are investigated and a similarity solution for laminar flow which depends on the eigenvalue problem $f'''+ff''+2f'^2=0$ with boundary conditions $f(0)=f'(0)=0$, $f'(\infty)=0$, is obtained. The velocity distribution is also determined. The turbulent wall jet is examined by introducing an eddy viscosity distribution near the wall, which is consistent with the Blasius profile and a constant eddy viscosity for the external part of the flow. The velocity distribution and values of the similarity exponents are obtained as functions of the Reynolds number. *H. Greenspan.*

Lew, H. G. On the stability of the axially symmetric laminar jet. Quart. Appl. Math. **13** (1955), 310–314.

The author investigates the stability of an axially symmetric laminar jet subjected to rotationally symmetric disturbances. By conventional methods it is shown that it is sufficient to consider only axially symmetric jets and that when viscosity is neglected there are no self excited or damped oscillations. *H. Greenspan.*

Nikitin, A. K. On the problem of steady motion of viscous incompressible fluid between pin and bearing. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 405–408. (Russian)

The problem of the title is reduced to an analysis of the well-known nonlinear equation

$$\nu\Delta\Delta\psi=\frac{\partial\Delta\psi}{\partial x}\frac{\partial\psi}{\partial y}-\frac{\partial\Delta\psi}{\partial y}\frac{\partial\psi}{\partial x}$$

for the stream-function, subjected to the boundary-conditions

$$\psi=0,\ \partial\psi/\partial n=-U \text{ on } C_1 \text{ (interior boundary)},$$
$$\psi=\text{const.},\ \partial\psi/\partial n=0 \text{ at } C_2 \text{ (contour of the encompassing boundary)}.$$

The author obtains a formal solution to this problem by means of successive approximations. The successive terms of the expansion are obtained from the solution of a certain bi-harmonic "Green's"-function. It is also shown that this process converges, provided certain inequalities are satisfied. A complete existence proof is, however, not presented. *K. Bhagwandin* (Oslo).

Krzywoblocki, M. Z. On the generalized fundamental equations for the interaction between dissipative flows and external streams. Acad. Serbe Sci. Publ. Inst. Math. **9** (1956), 9–39.

The problem of interaction between a viscous or dissipative flow near the surface of solid body, or in its wake, and an "outer" isentropic or nearly isentropic stream became very important. The author remodifies the Crocco-Lees approach [J. Aero. Sci. **19** (1952), 649–676] in order to include the rarefaction of the gas, the vertical velocity component and the pressure gradient in the vertical direction. The author uses Grad's equations based upon the kinetic theory of monatomic gases and preserves all the equations of momentum and energy in their full forms. A method, suitable for high-speed computing machines, is developed, enabling one to calculate the velocity, density, pressure and temperature distributions in the boundary layer, as well as the components of the the shearing stress tensor and heat flux vector. The convergence of the successive approximations process is proved; some remarks on the possible extension of the Crocco-Lees results in wakes to hypersonic flow regime close the paper. (Author's summary.) *J. B. Serrin.*

Ferrari, Carlo. Sullo strato limite laminare in corrente ipersonica. Aerotecnica **36** (1956), 68–94.

This paper deals with laminar boundary layers along curved insulated walls at hypersonic speeds, on the assumption that the product of the slope and of the free stream Mach number is small everywhere, while the Prandtl number is supposed to be equal to one. The solution depends on the familiar division of the boundary layer into outer and inner regions. Especially in the outer regions the calculations are extremely involved. For the particular case of a flat plate, the results are compared with some experimental data and satisfactory agreement is obtained. *A. Robinson* (Toronto, Ont.).

Tanaev, A. A. Influence of free convection on the coefficient of resistance of a plate for a laminar flow in the boundary layer. Ž. Tehn. Fiz. **26** (1956), 2563–2569. (Russian)

Taking gravity into account the author reduces the study of steady plane flow about a flat plate inclined at an angle α to the horizontal to consideration of

$$(*)\qquad \partial z/\partial \varepsilon = u_{cp}\partial^2 z/\partial\psi^2 + (Gr/Re^2)(\Theta_1+\Delta\Theta)\cos\alpha,$$

with $z(0,\psi)=0$; $z(\varepsilon,0)=\frac{1}{2}$; and $z(\varepsilon,\infty)=0$. Here ε is a horizontal coordinate, u horizontal velocity component, u_{cp} a mean value of u, $z=\frac{1}{2}(1-u^2)$, ψ a stream function, $Gr(Re)$ Grasshoff's (Reynolds's) number,

$$\Theta=(i-i_\infty)/(i_w-i_\infty),$$

and $i=c_pT$. Let $\zeta=\psi\varepsilon^{-\frac{1}{2}}$. Then Θ_1, which corresponds to a solution in which gravity has been neglected, is approximated by $\Theta_1=1-a\zeta^{1/n}$, $0\leqq\zeta\leqq a^{-n}$, and $\Theta_1=0$ for $\zeta\geqq a^{-n}$, where a and n are constants. If the correction term involving $\Delta\Theta$ in (*) is assumed to be negligible, then the boundary value problem (*) can be solved by means of Laplace transforms and by use of incomplete Gamma functions to obtain approximate skin friction and drag coefficients. As far as drag coefficients are concerned, the results show that gravity may be neglected if

$$Gr\cos\alpha/Re^2\leqq 0.1.$$

J. H. Giese (Aberdeen, Md.).

Squire, L. C. The three-dimensional boundary-layer equations and some power series solutions. Aero. Res. Council, Rep. and Memo. no. 3006 (1955), 17 pp. (1957).

The author derives the boundary-layer equations for a more general coordinate system than the orthogonal system used by Howarth [Phil. Mag. (7) **42** (1951), 239–243; MR **12**, 871]. In the new system, which is orthogonal only on the body surface, the coordinate lines on the body are not restricted to being lines of principal curvature. It is shown that if a streamline of the external flow evaluated at the body surface is a geodesic of the surface, the corresponding boundary-layer flow is in the same direction (i.e., there is no cross-flow inside the boundary layer). The solution of such a flow is found by a power series method. Finally, Howarth's stagnation-point solution [ibid. **42** (1951), 1433–1440; MR **13**, 505] is extended to second-order terms. Detailed tables of numerical values are presented. *D. W. Dunn* (Ottawa, Ont.).

Martin, John C. On Magnus effects caused by the boundary-layer displacement thickness on bodies of revolution at small angles of attack. J. Aero. Sci. **24** (1957), 421–429.

Es wird die laminare Grenzschicht eines rotierenden Zylinders, der mit einem kleinen Schiebewinkel angeblasen wird, für die Strömung inkompressibler Medien untersucht. Für die Strömung ausserhalb der Grenzschicht wird dabei angenommen, dass der Druckgradient in Richtung der Zylinderachse verschwindet. Für die Druckverteilung in Umfangsrichtung werden die von H. J. Allen [NACA Tech. Note no. **2044** (1950)] für die Umströmung schlanker, angestellter Körper angegebenen Beziehungen verwandt. Die zu dieser Aussenströmung gehörige dreidimensionale Grenzschicht wird exakt ermittelt. Die sich dabei für die Bestimmung der Geschwindigkeitsverteilung innerhalb der Grenzschicht als Lösung der Diff. Gl. ergebenden Funktionen werden in der vorliegenden Veröffentlichung nicht numerisch mitgeteilt, es wird auf den Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. **870** (1955) des gleichen Verfassers verwiesen. Die Verdrängungsdicke dieser dreidimensionalen Grenzschicht ist unsymmetrisch zur Schiebeebene, so dass aus der dadurch veränderten Potentialströmung eine Kraft senkrecht zur Schiebeebene folgt, die mit Magnus Effekt in einer verallgemeinerten Definition bezeichnet wird. Für den Magnus Effekt wird auch eine Abschätzung für turbulente Grenzschichten angegeben. *L. Speidel* (Mülheim).

Sedney, R. Laminar boundary layer on a spinning cone at small angles of attack in a supersonic flow. J. Aero. Sci. **24** (1957), 430–436, 455.

Es wird die laminare Grenzschicht eines rotierenden

mit kleinem Schiebewinkel angeblasenen Kegels für den Überschallbereich ermittelt. Dabei wird für die Prandtl-Zahl der Wert $Pr=1$ und für die Zähigkeit eine lineare Funktion der Temperatur angenommen. Die Lösung der Grenzschichtgleichungen wird als Potenzreihe der kleinen Veränderlichen Schiebewinkel α und Rotationsparameter κ dargestellt, wobei nur die Glieder 1. Ordnung $\alpha\cdot\kappa$, die die Wechselwirkung zwischen Schieben und Drehung beschreiben, Berücksichtigung finden. Die Lösung der Grenzschichtgleichungen wird dann exakt ermittelt, die Funktionswerte werden aber nicht numerisch mitgeteilt, es wird auf den Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. **991** verwiesen. Die Wandschubspannung sowie die Verdrängungsdicke der Grenzschicht wird bestimmt. Aus dem zur Schiebeebene unsymmetrischen Verlauf der Verdrängungsdicke wird abschliessend der Magnus-Effekt berechnet [s.a. J. C. Martin, ibid. no. **870** (1955)]. *L. Speidel* (Mülheim).

Tchen, Chan-Mou. Approximate theory on the stability of interfacial waves between two streams. J. Appl. Phys. **27** (1956), 1533–1536.

The author studies the stability of the interface of two superposed streams considering the effects of viscosity, gravitational acceleration and surface tension. The basic flow consists of two parallel uniform streams, which can exist only when the effect of viscosity is neglected. The author gave some but not a complete discussion of the basis for this kind of approximation. Under his assumptions, approximate results are obtained and discussed from the physical point of view. *C. C. Lin.*

Di Prima, R. C.; and Dunn, D. W. The effect of heating and cooling on the stability of the boundary-layer flow of a liquid over a curved surface. J. Aero. Sci. **23** (1956), 913–916.

"The theory of the three-diemnsional instability of laminar boundary layers over curved concave surfaces is extended to include the effects of heat transfer. Results of numerical calculations for the boundary-layer flow of water indicate that heating and cooling of the wall have only a small influence on this type of instability compared to their effect on the usual two-dimension instability". (Authors' summary.) *C. C. Lin* (Cambridge, Mass.).

Szablewski, W. Turbulente Vermischung ebener Heissluftstrahlen. Ing.-Arch. **25** (1957), 10–25.

On the basis of similarity, the problem of a heated free jet was investigated in a manner quite analogous to the incompressible case. Differential equations were integrated by iteration for different velocity- and temperature-differences. By a suitable choice of parameter, the results obtained are compared favourably with experiments. *Y. H. Kuo* (Peking).

Craya, A. Idées actuelles sur les mécanismes de la turbulence et des transferts thermiques turbulents. Houille Blanche **12** (1957), 19–28.

The paper stresses the inadequacy of older concepts on the mechanism of turbulence and attempts to synthesize the developments of the last ten years. The paper is based principally on the ideas advanced by Townsend. *J. Kestin* (Providence, R.I.).

Ovsyannikov, L. V. A new solution of the equations of hydrodynamics. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 47–49. (Russian)

Author finds a new exact, particular solution of the equations of adiabatic motion of a compressible fluid in matrix form. *M. D. Friedman* (Newtonville, Mass.).

Keune, Friedrich. Grundsätzliche Betrachtungen zur Unter- und Überschallströmung um Körper nicht mehr kleiner Streckung. Z. Flugwiss. **5** (1957), 121–125.

If slender-body theory is taken as a first approximation, for small aspect ratio, say, to the solution of the linearized small-perturbation equations, one may seek an improved approximation for what Adams and Sears called "not-so-slender" bodies and/or wings [J. Aero. Sci. **20** (1953), 85–98; MR **14**, 599]. The present author argues that the next terms must come from the higher moments of the source distribution in each cross section. For configurations with lateral symmetry the first moment vanishes and the next contribution after slender-body theory must come from the second moment. He states that this conclusion is confirmed by his published work [Roy. Inst. Tech., Div. Aero., Stockholm, KTH-Aero. Tech. Note no. **21** (1952)], and can be drawn from the paper by Adams and Sears mentioned above. Various consequences regarding wave drag and other matters are stated. The present paper is a summary; the details of the analysis are to appear in a subsequent paper. *W. R. Sears* (Ithaca, N.Y.).

Berndt, Sune B. On the drag of slender bodies at sonic speed. Flygtekn. Försöksanstalt. Rep. **70** (1956), 17 pp.

This is an interesting theoretical discussion of the drag of slender bodies in the neighborhood of the velocity of sound. A formula for the drag is obtained by momentum considerations. Use is made of the fact that linearised theory alone is sufficient to correlate the fields of flow due to two different bodies with equal distribution of cross-sectional area. Since linearised theory at Mach number one reduces the problem to the solution of Laplace's equation in the transverse planes, conformal mapping methods are applicable. There result rules for obtaining bodies of revolution with minimum drag (or maximum drag) for transonic speeds. While the results are known from linearised theory, the present analysis widens the range of their applicability. *A. Robinson.*

Bader, W. Iteratives Näherungsverfahren zur Druckbestimmung bei stationärer ebener Unterschallströmung. Z. Angew. Math. Mech. **36** (1956), 296.

Choose a lattice of points at the intersections of discrete sets of approximate streamlines and approximate equipotential lines. Determine an approximate velocity distribution on the lattice by means of the equations of continuity and irrotationality. Then improve the streamlines and equipotential lines and repeat. Details will be given in a forthcoming paper. *J. H. Giese.*

Runyan, Harry L.; and Woolston, Donald S. Method for calculating the aerodynamic loading on an oscillating finite wing in subsonic and sonic flow. NACA Tech. Note no. **3694** (1956), 76 pp.

The integral equation for this case was discussed by Watkins, Runyan, and Woolston in an earlier report [NACA Tech. Note no. **3131** (1954); NACA Rep. no. **1234** (1955); MR **15**, 474]. It constitutes an expression for the downwash distribution in terms of the loading (pressure). Here its solution is undertaken in series form. The loading over the wing is assumed in the form of a sum of convenient elementary loadings. In view of the complexity of the kernel, however, the integrations of these element-

ary loadings in the chordwise direction are not actually carried out, but these loadings are approximated once more by "replacement loads" which produce the same total load and the same downwash at selected points of the planform. An example involving a rectangular wing oscillating rigidly in pitching is carried out in detail. In another example, the method of calculating the pitching moment is indicated.

Finally, a multiple-lifting-line approximation based on the steady-flow method of Schlichting and Kahlert [Roy. Aircraft Establishment, Rep. No. Aero. **2297** (1948)] is worked out and is applied to the rectangular-wing problem mentioned above. The results are compared with those of the lifting-surface technique developed here, over a range of reduced frequencies. Agreement seems good when four lifting lines are used. Other comparisons with results of Lawrence and Gerber [J. Aero. Sci. **19** (1952), 769–781, **20** (1953), 296] and with experimental results are carried out. *W. R. Sears* (Ithaca, N.Y.).

Kusukawa, Ken-ichi. On the transonic flow of a compressible fluid past an axisymmetric slender body at zero incidence. J. Phys. Soc. Japan **12** (1957), 401–410.

After being subjected to an affine transformation and the neglect of one term, the differential equation for the stream function of axisymmetric transonic small-perturbation flow is formally the same as that for plane transonic flow. The boundary conditions, however, are not the same. Here the respective transonic similarity laws for plane and axisymmetric flow are drawn upon to obtain an approximate relation between the velocities at the surfaces of the obstacles. An example is presented in which the results for a finite wedge at sonic speed, obtained by Guderley and Yoshira [J. Aero. Sci. **17** (1950), 723–735; MR **15**, 264], are used to obtain the pressure distribution for a paraboloid of revolution. An attempt is also made to relate the head drag coefficient given by the present theory to that obtained by Miles [ibid. **23** (1956), 146–154; MR **17**, 799], and it is concluded that the Miles formula probably fails for blunt obstacles. *W. R. Sears.*

Kusukawa, Ken-ichi. On the transonic flow of a compressible fluid past a nearly axisymmetric slender body. J. Phys. Soc. Japan **12** (1957), 411–419.

For nearly axisymmetric bodies, the perturbation flow pattern is divided into two parts. The first of these is axisymmetric and is treated by the method of the paper reviewed above. The other part is governed by the two-dimensional Laplace equation in planes normal to the stream; this can be calculated by standard methods. Two examples are presented: (i) an elliptic paraboloid at zero incidence, (ii) a paraboloid of revolution at small incidence. *W. R. Sears* (Ithaca, N.Y.).

Cole, J. D. Newtonian flow theory for slender bodies. J. Aero. Sci. **24** (1957), 448–455.

As an aid to the aerodynamicist in the design of air frames for hypersonic speeds (speeds faster than about Mach 5), Newtonian flow theory is examined from the point of view of gas dynamics and hypersonic small-disturbance theory. The usual theory is shown to result as the first approximation of an expansion valid for small $\lambda=(\gamma-1)/(\gamma+1)$. A basic similarity parameter

$$N=(\gamma+1)/(\gamma-1)M_\infty{}^2\delta^2$$

is introduced. A general solution of the first approximation for the flow past slender bodies (bodies which cause only a small disturbance to the stream) at zero angle of attack is given. An important condition which limits the application of the theory is noted – namely, that the pressure coefficient on the surface not fall to zero. The theory is then applied to cones and to bodies whose shape is $r=x^n$. *Author's summary.*

Powell, J. B. L. The effect of dihedral on the aerodynamic derivatives with respect to sideslip for airfoils in supersonic flow. Quart. J. Mech. Appl. Math. **9** (1956), 425–440.

Continuing earlier work [same J. **9** (1956), 51–74; MR **17**, 914] the author investigates the effect of sideslip on wings with dihedral. A cone field method is used to solve the problem and the appropriate aerodynamic derivatives are calculated. For small dihedral there is agreement with the results of A. Robinson and J. H. Hunter-Tod [Coll. Aero. Cranfield Rep. no. **12** (1947); MR **9**, 479] and, for moderate dihedral, with the results of S. Nocilla [Aerotecnica **34** (1954), 126–141; MR **16**, 642].

The author's statement that Robinson and Hunter-Tod used a source distribution method is misleading since in actual fact this applies only to the definitely supersonic case. A cone-field procedure was used for the quasi-subsonic case. *A. Robinson* (Toronto, Ont.).

Vallée, D. Effet de dérapage sur une aile delta. Rech. Aéro. no. **57** (1957), 3–6.

Le travail présenté dans cet article a constitué une partie d'une communication faite au IXe Congrès de Mécanique appliquée de Bruxelles. Il est consacré à l'étude du dérapage, quelconque en grandeur, sur une aile delta sans incidence et présentant un léger dièdre. L'écoulement est supposé stationnaire et supersonique et l'aile intérieure au cône de Mach. Les efforts calculés sont la portance, la traînée, les moments de tangage et de roulis. *Résumé de l'auteur.*

Guienne, Paul; et Bouniol, Fernand. Détermination du champ de vitesses en aval d'un choc détaché. C. R. Acad. Sci. Paris **243** (1956), 1479–1482.

Les auteurs déterminent les lignes de courant et les nombres de Mach en aval d'une onde de choc détachée dans un écoulement plan, à partir de la mesure de la messe spécifique obtenue par interferométrie. A partir de l'onde de choc, le calcul est éffectué par mailles, limitées aux lignes de courant sur deux cotés; l'entropie est ainsi connue, au lieu d'être déterminée par approximations, comme dans le méthode des caractéristiques. A partir d'une ligne de courant supersonique, on obtient, à l'aide de l'équation de continuité, les lignes de courant dans le domaine subsonique. *H. Cabannes* (Marseille).

Heinz, C. Reflexion ebener Druckwellen an einer festen Wand. Z. Angew. Math. Mech. **37** (1957), 63–73. (English, French and Russian summaries)

Es wird ein einfaches Verfahren angegeben, um die Reflexion einer aus einem Verdichtungsstoß und einer nachfolgenden Verdünnungswelle bestehenden ebenen Druckwelle an einer festen Wand zu berechnen. Hierbei werden die von Hadamard angegebenen Lösungen der Darbouxschen Gleichung, die bei dem hier behandelten Problem als Potentialgleichung in der Geschwindigkeitsebene auftritt, in iterativer Form zur gleichzeitigen Bestimmung des Stoßes und des Strömungsfeldes zwischen dem Stoß und der Wand benutzt. Auch die inverse

Aufgabe, bei der der Druck an der Wand gegeben und daraus die ankommende Welle zu berechnen ist, wird behandelt. Als Beispiel wird die Reflexion einiger Wellen durchgerechnet. Ein Vergleich mit dem Ergebnis der wesentlich umständlicheren Charakteristikentheorie ergibt keinen merklichen Unterschied.
Zusammenfassung des Autors.

Müller, E.-A. Das zeitliche Abklingen der Störungen nach der Umlenkung eines fortschreitenden Verdichtungsstosses durch einen schwachen Knick in einem Kanal konstanten Querschnitts. Z. Flugwiss. **5** (1957), 114–120.

Das räumlich zweidimensionale, instationäre Problem wird durch Linearisierung der reibungslosen gasdynamischen Grundgleichungen und Superposition geeigneter „kegeliger" Grundlösungen für beliebig starke Verdichtungsstöße behandelt. Für die Zustandsgrößen ergeben sich endliche Summenausdrücke. Aus diesen wird für große Zeiten ein analytischer Ausdrück für die „Einstellzeit" gewonnen (d.i. die Zeit, bei der sich der neue Zustand am Verdichtungsstoß bis auf eine bestimmte Abweichung eingestellt hat). Die Abhängigkeit der Ergebnisse von der Machschen Zahl des Verdichtungsstoßes ($1<M<5$) wird diskutiert. *Zusammenfassung des Autors.*

Lyamšev, L. M. Diffraction of sound on a thin bounded plate in a fluid. Akust. Ž. **1** (1955), 138–143. (Russian)

The problem treated is the diffraction of sound waves by an infinite strip of thin elastic material serving as a window in an infinite, plane, rigid screen. Both bending and compression of the elastic window are taken into account. The velocity distribution on the plate is expanded in a series of eigenfunctions obtained by considering the free vibrations of the plate, in absence of any fluid. The problem is thus reduced to an infinite set of equations in an infinite number of unknowns. This set is assumed diagonal when the plate is many wavelengths wide. The author finds substantial scattering in the direction opposite to the direction of incidence. *J. Shmoys.*

Tayler, R. J. A note on hydromagnetic stability problems. Phil. Mag. (8) **2** (1957), 33–36.

For normal modes with time-dependence $e^{i\omega t}$ it may happen that ω is not uniquely determined by the remaining propagation constants. To fix the latter by putting $\omega=0$ will in such cases not necessarily yield cases of marginal stability, since there may also be imaginary ω corresponding to the same propagation constants. A numerical example is given in support, being a special case of an investigation of J. W. Dungey and R. E. Loughhead [Austral. J. Phys. **7** (1954), 5–13; MR **15**, 1008; see also Loughhead, ibid. **8** (1955), 319–328; MR **17**, 561]. His basic equation (3) corrects a misprint in the first-cited paper, but commits another.
F. V. Atkinson (Canberra).

Janossy, L. Generalized form of the diffusion equation for a single particle. Soviet Physics. JETP **3** (1956), 315–322.

English translation of article first published in Z. Eksper. Teoret. Fiz. **30** (1956), 351–361, [MR **18**, 76].

See also: Kreĭn, p. 36; Godunov, p. 37; Morawetz, p. 40; Crabtree and Woollett, p. 65; Crank, p. 66; Maslen and Ostrach, p. 95; Hellman, p. 95.

Optics, Electromagnetic Theory, Circuits

Focke, Joachim. Der Einfluss des Öffnungsfehlers auf die Bildgüte. Opt. Acta **4** (1957), 17–21.

The wave-optical image of a half plane, formed by an optical system with spherical aberration, is investigated. The distribution of the light intensity is obtained by adding the light distributions of the separate points of the illuminated half plane. These can be evaluated numerically, or — if the spherical aberration predominates over the diffraction effects — by means of an approximate formula. After the image of a half plane has been found, the images of strips of various widths and of gratings are easily obtained. The results are illustrated by curves representing the intensity distribution in a number of cases. *N. G. van Kampen* (Utrecht).

Haus, H. A.; and Bobroff, D. L. Small signal power theorem for electron beams. J. Appl. Phys. **28** (1957), 694–704.

The interaction between an electron beam and the microwave structure is investigated for electron beam tubes. The basic equations of the electron-field interaction are linearized by the small signal assumption. It is shown that this assumption implies a conservation theorem for any general beam. Hence all microwave amplifiers analyzed under it may be treated as lossless linear networks provided the beam is included in the analysis. Equations are derived for a filament beam in arbitrary dc electric and magnetic fields. The small signal ac power theorem is given for a two-dimensional rectilinear beam of finite thickness. Applications to the noise theory and signal analysis of electron beams are indicated. *J. E. Rosenthal* (Passaic, N.J.).

Bernard, Michel-Yves; et Hue, Jean. Les aberrations géométriques dans les lentilles à focalisation forte. C. R. Acad. Sci. Paris **243** (1956), 1852–1854.

In this note the authors have established the equations of the trajectories of charged particles for strong focusing lenses. The electrostatic and magnetic rotational fields are expanded up to fourth degree terms in (x, y) with coefficients depending on the axial coordinate z of the lens system. The equations in the deviation image functions depend on the Gaussian image functions. They are non-homogeneous and of the second order. The non-homogeneous terms are limited to third degree terms in the Gaussian image functions. To this order of approximation the geometrical aberration effects are of third order and they arise mainly from the tailing off of the fields (electrostatic case). Due to the quadruple symmetry of the system, the aberration coefficients are reduced from 80 to 40 in number. They have been numerically calculated by an electronic machine with the help of measured data of the field distribution. *N. Chako* (New York, N.Y.).

Tonolo, Angelo. Sulla determinazione del campo elettromagnetico all'interno di un conduttore omogeneo e isotropo. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 556–560.

This is a continuation of part I [same Rend. (8) **20** (1956), 403–408; MR **18**, 847]. *E. T. Copson* (St. Andrews).

Gershman, B. N. Note on waves in a homogeneous magnetoactive plasma. Soviet Physics. JETP **4** (1957), 582–585.

If one looks for plane wave solutions of the linearized

equations of motion for a 'quasi-hydrodynamical' plasma in a constant magnetic field, a sixth-order dispersion equation is obtained. The three pairs of roots correspond to three kinds of high-frequency waves: ordinary, extraordinary and plasma waves. It is here emphasized that this separation is not fundamental, because when the parameters in the equation (either the temperature or the angle between the direction of propagation with the magnetic field) are varied, the three pairs of roots go over into each other continuously. It is also pointed out that a more detailed kinetic treatment leads to similar equations, in which, however, damping effects are included.

N. G. van Kampen (Utrecht).

Bass, F. G.; and Tsidil'kovskii, I. M. Theory of isothermal galvanomagnetic and thermomagnetic effects in semiconductors. Soviet Physics. JETP 4 (1957), 565–574.

An effective method of investigating the properties of semiconductors is the study of galvanomagnetic and thermomagnetic effects. The previous theories of these effects refer to weak magnetic fields. Bass and Tsidil'kovskii extend the theory to intermediate and strong magnetic fields for various types of interaction of the carriers with the crystal lattice. Starting from the kinetic equations for the distribution function of the carriers in coordinate-momentum space, they derive the asymptotic formulas for the electric current density and the density of heat current, which formulas are limited to very large and very small magnetic fields. Using these, the writers consider the galvanomagnetic effects and calculate the two components of the electric field. It appears that the Hall field, in both limiting cases, varies linearly with the effective magnetic field. Particular cases discussed are: semiconductor with an atomic lattice; with ionic lattice at temperatures below and above the characteristic temperature. Next, the thermomagnetic effects are considered showing that the relative change of the coefficient of heat conductivity is a quadratic function of magnetic field in weak magnetic fields and approaches unity in strong fields. The same pattern of reasoning is applied to semiconductors with mixed conduction; the electric and heat currents are defined as the sums of the electron current and the hole current. In this case the thermomagnetic effects show that in strong effective magnetic fields the dependance of the components of the electric field upon the magnetic field is different for different ratios between the concentrations of the electrons and of the holes. *M. Z. Krzywoblocki* (Urbana, Ill.).

Lowndes, J. S. A transient magnetic dipole source above a two-layer earth. Quart. J. Mech. Appl. Math. 10 (1957), 79–89.

A magnetic dipole with its axis parallel to the z axis acting as a transient source is located at a point $(0, 0, a)$. The space is divided into three regions $(0\leqq z<\infty)$, $(0\geqq z\geqq -h)$, $(-h\geqq z>-\infty)$ with the respective characteristics $(\sigma_1, \varepsilon_1, \mu_1)$, $(\sigma_2, \varepsilon_2, \mu_2)$, $(\sigma_3, \varepsilon_3, \mu_3)$ which denote respectively the electrical conductivity, dielectric constant and permeability. Expressions for the induced electromagnetic field components are deduced using the method of integral transforms (Laplace and Hankel transforms). The following special cases are evaluated. A magnetic dipole acting (1) as a step function current, (2) as an impulsive current source over a homogeneous and over a two layer earth.

F. Oberhettinger.

Pipes, Louis A. Matrix theory of skin effect in laminations. J. Franklin Inst. 262 (1956), 127–138.

An alternating magnetic field is impressed upon plane conducting metal plates of constant permeability. The magnetic and electric field intensities and the density of the induced currents are represented in matrix form. It is demonstrated in some examples that this technique may lead to a simplification of the calculations involving skin effect and magnetic shielding.

F. Oberhettinger.

Halfin, L. A. The field of a point source in the presence of a semispheroidal cavity. Izv. Akad. Nauk SSSR. Ser. Geofiz. 1956, 1200–1206. (Russian)

Applying the results and formulae of his previous work [same Izv. **1956**, 657–668; MR **18**, 265] to the case of a point-source located on the ground $(x=0)$ near to a semipheroidal depression for which the plane $x=0$ is a plane of symmetry, the author obtains the expansions into series of spheroidal harmonics for the gradient of potential on some rectilinear profiles.

E. Kogbetliantz.

Williams, W. E. A note on the diffraction of a dipole field by a half-plane. Quart. J. Mech. Appl. Math. 10 (1957), 210–213.

T. B. A. Senior [same J. **6** (1953), 101–114; MR **14**, 933], Y. V. Vandakurov [Ž. Eksper. Teoret. Fiz. **26** (1954), 3–18; MR **16**, 884] and the reviewer [I. R. E. Trans. **AP-4** (1956), 294–296] have solved the problem of the diffraction of a dipole field by a perfectly conducting lpane. The author solves the same problem in a more "compact" form, but there are some details which appear to need further discussion. *A. E. Heins.*

Jones, D. S. High-frequency scattering of electromagnetic waves. Proc. Roy. Soc. London. Ser. A. 240 (1957), 206–213.

The assumption that the illuminated region and the penumbra of a body scatter independently at high frequencies is used to obtain scattering coefficients for perfectly reflecting convex bodies in plane electromagnetic and sound waves. The formulas involve only the scattering coefficients of the circular cylinder and the geometry of the shadow boundary.

One general result is that the electromagnetic scattering coefficient of a solid of revolution, when the direction of propagation of the incident wave is along the axis of revolution, is the average of the sound-hard and sound-soft scattering coefficients. *A. E. Heins.*

Avazašvili, D. Z. The problem of diffraction in a multiply connected region. Dokl. Akad. Nauk SSSR (N.S.) 110 (1956), 889–892. (Russian)

Bremmer, H. Diffraction problems of microwave optics. I. R. E. Trans. AP-3 (1955), 222–228.

This is a status report in which the author surveys mathematical methods currently applied in the diffraction theory of microwaves. It is mainly a review of papers presented at the Symposium on Microwave Optics held at McGill University, Montreal, Canada, June 22–25, 1953. *C. J. Bouwkamp* (Eindhoven).

Goryainov, A. S. Diffraction of a plane electromagnetic wave by a conducting cylinder. Dokl. Akad. Nauk SSSR (N.S.) 109 (1956), 477–480. (Russian)

Burštein, E. L.; and Solov'ev, L. S. On the diffraction of a finite beam of electromagnetic waves upon a cylindrical obstacle. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 473–476. (Russian)

Lenoble, Jacqueline. Etude théorique de la pénétration du rayonnement dans les milieux diffusants naturels. Opt. Acta 4 (1957), 1–11.

The author shows that it is possible by using the method of Chandrasekhar [Radiative transfer, Oxford, 1950; MR 13, 136] to calculate to a sufficient approximation the penetration of light into such diffusing media as a layer of fog or the sea. She points out that a precision of about 10 per cent is possible even by using hand calculating machines and such a precision is probably sufficient in view of the uncertainty of atmospheric optical data. The calculation fails in the neighborhood of the surface of the diffusing medium, as would be expected. An interesting point that emerges is that the shape of the diffusion function can vary widely without very much effect on the final distribution of luminance in various directions.

W. E. K. Middleton (Ottawa, Ont.).

Burberg, Rudolf. Die Lichtstreuung an kugel- und stäbchenförmigen Teilchen von Wellenlängengrösse. Z. Naturf. **11a** (1956), 807–819.

Papadopoulos, V. M. The scattering effect of a junction between two circular waveguides. Quart. J. Mech. Appl. Math. **10** (1957), 191–209.

The reflection of the dominant E mode at a discontinuity of cross section in a circular wave guide is calculated on the assumption that only this mode can be sustained on both sides of the discontinuity. A Wiener-Hopf integral equation governing the annular discontinuity of magnetic field is established; its solution leads to an infinite set of algebraic equations. These are solved approximately on the assumption that the change in radius is small and numerical results for the reflection and transmission coefficients plotted. This method of attack complements the matching of field components across the aperture, which is better suited to relatively large changes of radius. [The reviewer notes that numerical results obtained by the latter method for the corresponding acoustic problem were given by Miles, J. Acoust. Soc. Amer. **22** (1950), 59–60.] *J. W. Miles* (Los Angeles, Calif.).

Ahiezer, A. I.; and Lyubarskiĭ, G. Ya. On the theory of coupled resonant cavities. II. Ž. Tehn. Fiz. **25** (1955), 1597–1603. (Russian)

[For part I see same Ž. **24** (1954), 1697–1706; MR **16**, 774.] The problem considered is that of propagation of waves in a chain of identical resonators coupled to each other by identical long and narrow slots. The author attacks the problem by setting up an integro-differential equation involving the field distributions in slots $(n-1)$, n, and $(n+1)$. These fields are then assumed to differ in phase only for a propagative wave. The solution is then obtained as a power series in $\alpha=\log(L/d)$, where L and d are the length and width of the slot; two terms of the series are considered. The author distinguishes between "slot waves", dependent primarily on slot resonance and "cavity waves" at frequencies close to cavity resonances. The case in which a slot resonant frequency and a cavity resonant frequency coincide is also discussed.

J. Shmoys (Brooklyn, N.Y.).

Lignon, Jacques R. Directivity of end fire arrays of isotropic antennas. An. Acad. Brasil. Ci. **28** (1956), 439–446.

The author derives an expression for the directivity of an end fire linear array of isotropic point sources of equal amplitude. This is evaluated for the case of ordinary directivity, and for the case of increased directivity [Hansen and Woodyard, Proc. I.R.E. **26** (1938), 333–345].

W. K. Saunders (Washington, D.C.).

Krishna Bhattacharyya, Bimal. Field on the earth's surface due to a transient electromagnetic disturbance. J. Tech. Bengal Engrg. Coll. **1** (1956), 151–162.

Formulae of the electric and magnetic fields above a homogeneous earth have been derived for a small loop antenna energised by transient currents, e.g., step- and ramp-function types. The mutual impedance function between a small length of wire and a small loop of antenna has been determined. A formula of the voltage induced in a secondary loop of area dA_1 due to the circulating current in another loop of area dA has also been deduced. The mutual impedance function $Z_m(t)$ and the induced voltage $v(t)$ have been plotted as a function of time for both step- and ramp-function current sources. The nature of variation of $Z_m(t)$ and $v(t)$ with the change in steepness of the pulse-fronts of the energising current has been discussed. *Author's summary.*

Braude, B. V. On an estimate of the error in using the integral-equation method in antenna theory. Ž. Tehn. Fiz. **25** (1955), 1819–1824. (Russian)

The paper deals with the effect of omitting the field due to a magnetic current ring along the gap in an antenna. The author finds that the effect is negligible if the antenna is thin, but that it is serious when the thickness is comparable to the wavelength. *J. Shmoys.*

Bashkow, T. R.; and Desoer, C. A. A network proof of a theorem on Hurwitz polynomials and its generalization. Quart. Appl. Math. **14** (1957), 423–426.

The paper refers to a canonical form of Hurwitz-polynomials, given by the reviewer in the form $f(p)=\det(c_{ik})$ with the matrix (c_{ik}) of triple diagonal form with p appearing in the main diagonal linearly, while the diagonal below has elements $+1$ and the diagonal above elements -1. The authors give an alternate proof by network theoretic methods by considering the Cauer form of a ladder network; they also obtain a similar canonical form by means of the Foster form of a ladder network.

H. Bückner (Schenectady, N.Y.).

Adams, K. M. On the synthesis of 3-terminal RC networks. Coll. Aero. Cranfield. Rep. no. **96** (1956), 43 pp.

This paper concerns 3-terminal RC network with given input, output, and transfer impedances. Necessary conditions for realizability are given.

A synthesis procedure is described which depends on the possibility of satisfying a certain inequality.

R. J. Duffin (Pittsburgh, Pa.).

Armstrong, H. L. On the connection between transmission matrices and Green's theorem. Matrix and Tensor Quart. **7** (1956), 11–15.

A function ϕ is to satisfy the equation $\nabla^2\phi+k^2\phi=0$ in a region R. On two boundary surfaces S_1 and S_2 of R, ϕ is to be constant and the normal derivative $\partial\phi/\partial n=0$ on the rest of the surface of R. Then by Green's theorem it is

shown that ϕ and $\int \partial\phi/\partial n\, dS$ evaluated on S_1 and S_2 give four quantities related by a two by two matrix. This matrix is similar to the transmission matrix of four-terminal network theory and thus network theory might be applied to a variety of physical situations.

R. J. Duffin (Pittsburgh, Pa.).

DeClaris, N. An existence theorem for driving-point impedance functions. J. Math. Phys. **35** (1956), 83–88.

Of concern are networks of RC type except for one inductance. It is shown that the driving point impedance will have at most one pair of complex conjugate zeros and one pair of complex conjugate poles.

If the real poles and zeros are arranged in order of magnitude, then no more than three zeros or three poles can occur in succession. *R. J. Duffin.*

Kuh, E. S. Special synthesis techniques for driving point impedance functions. I. R. E. Trans. on Circuit Theory **CT-2** (1955), 302–308.

Of concern is the synthesis of impedance by an RLC network not having mutual inductance. The Bott-Duffin method will do this but is not economical of network elements. This paper is an extension of the method of F. Miyata [J. Inst. Elec. Comm. Engrs. Japan **35** (1942), 211–218]. The method of Miyata does not always work but when it does, it is more economical that the Boot-Duffin method. The impedance function is classified by the location of the zeros of its real part. Various techniques to carry out the synthesis are illustrated.

R. J. Duffin (Pittsburgh, Pa.).

★**Guillemin, E. A. New methods of driving-point and transfer impedance synthesis.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 119–144. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1956.

This paper is concerned with the extension of the Miyata synthesis method as is the paper of Kuh reviewed above. Also a quasi-Darlington synthesis procedure is developed. It is pointed out that while these methods may not always apply, they may be easier to compute than the Bott-Duffin method. *R. J. Duffin* (Pittsburgh, Pa.).

Lafleur, Charles. Sur une partie réelle associée à l'impédance d'un circuit purement réactif. C. R. Acad. Sci. Paris **243** (1956), 645–647.

It is usually supposed that the real part of a reactive impedance vanishes on the imaginary axis. The author advances reasons for treating the real part as a Schwartz distribution involving Dirac delta functions.

R. J. Duffin (Pittsburgh, Pa.).

★**Piloty, H. Transfer-matrix of reciprocal passive two-ports with prescribed transfer and reflection-coefficients.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 349–360. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1956.

Transfer and reflection coefficients are defined for a two-port working between ohmic resistance. Necessary and sufficient conditions are stated for the behavior of these coefficients as a function of the frequency.

R. J. Duffin (Pittsburgh, Pa.).

Sved, G. An electrical resistance network analogue for the solution of moment distribution problems. Austral. J. Appl. Sci. **7** (1956), 199–204.

The moments in multistory building frames are usually analyzed by the moment distribution method of Hardy Cross. This paper gives a network analog to this method in which currents correspond to moments.

R. J. Duffin (Pittsburgh, Pa.).

★**Gaponov, A. V. Electromechanical systems with sliding contacts and the dynamical theory of electrical machines.** Pamyati Aleksandra Aleksandroviča Andronova [In memory of Aleksandr Aleksandrovič Andronov], pp. 196–214. Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.

★**Грузов, Л. Н. [Gruzov, L. N.] Методы математического исследования электрических машин. Metody matematičeskogo issledovaniya èlektriceskih mašin. [Methods for the mathematical investigation of electrical machines.]** Gosudarstv. Ènerget. Izdat., Moscow-Leningrad, 1953. 264 pp. (1 plate). 10.25 rubles.

This is essentially a mathematical treatise on the theory of rotating electrical machines. It is addressed to the specialist rather than the student, and, in effect, assumes a familiarity with vector analysis, matrix algebra and operational calculus.

The principal problems treated in this text are the following: analysis of multi-winding transformers, analysis of synchronous and asynchronous machines, behavior of tandem and parallel combinations of machines, oscillatory and transient phenomena. The author's treatment of these problems follows the lines of Kron's work [Gen. Elec. Rev. **38** (1935), 181–191, 230–243, 282–292, 339–344, 386–391, 434–440, 473–479, 527–536, 582–591; **39** (1936), 108–116, 155–159, 201–210, 249–257, 297–306, 397–402, 504–509; **40** (1937), 101–107, 197–202, 296–302, 389–396, 490–496, 594–601; **41** (1938), 153–159, 244–250, 448–454] but is free of the unnecessary conceptual apparatus which made Kron's work so difficult to understand. There are numerous references to the work of Soviet investigators and many of the analyses are credited to them. *L. A. Zadeh.*

See also: Egorov, p. 23; Kontorovič, p. 32; Kron, p. 64; Longuet-Higgins, p. 76; Shabanskii, p. 78; Haskins and Walsh, p. 83; Tayler, p. 91; Ovsyannikov, p. 99; Logunov, p. 99; Tauber, p. 103; Schmutzer, p. 103; Prihar, p. 107.

Classical Thermodynamics, Heat Transfer

★**Vernotte, Pierre. Calcul numérique, calcul physique, application à la thermocinétique. Préface de G. Ribaud.** Publ. Sci. Tech. Ministère de l'Air, Paris, no. 319, 1956. xi+344 pp. 2200 francs.

L'objet de ce livre est le résolution mathématique de nombreux problèmes relatifs aux transferts de chaleur. La première partie est un exposé des théories thermocinétiques fondamentales: problème du mur, de la barre, différents âges du refroidissement, ondes thermiques. Le seconde partie est consacrée aux méthodes de calcul numérique: sommation des séries et exploitation des données experimentales (lissage des courbes, intégration et dérivation empiriques). Une grande place est accordée à la terminologie; il nous est rappelé judicieusement que la chaleur ne se propage pas, car la propagation n'est vraiment authentique que s'il s'agit de la Foi. L'ouvrage

se termine par 26 pages constituées par le reproduction de publications antérieures de l'auteur. *H. Cabannes.*

Maslen, Stephen H.; and Ostrach, Simon. Note on the aerodynamic heating of an oscillating insulated surface. Quart. Appl. Math. **15** (1957), 98–101.

This very interesting paper contains a study of the thermal boundary layer on an oscillating, infinite flat plate when the properties of the surface are suddenly changed from being conducting to becoming adiabatic. Thus at time $t=t_0$ the boundary condition changes from $T_s(0, t)=T_0-U^2Pr/4cp$ to $T_y(0, t)=0$ for $t>t_0$. Here $T(y, t)$ is the temperature in the fluid at right angles, y, to the plate, and the remaining symbols are self-explanatory.

Expansions are derived for the velocity profile $T(y, t)$ and for the recovery factor r. The latter is also given in a form averaged over one cycle, $\bar{r}$. The recovery ratio $r_1=\bar{r}/r_s$, where $r_s=Pr^{\frac{1}{2}}$, can be either less or greater than unity. Factors $r_1<1$ persist only for a short time.

J. Kestin (Providence, R.I.).

Hellman, O. A special problem of compressible fluid flow in ducts with friction and heat addition. Ann. Acad. Sci. Fenn. Ser. A. I. no. **231** (1957), 16 pp.

The paper gives an approximate series solution for the problem of heat transfer between a tube and a compressible stream in parallel flow. The fundamental equations are cast in accordance with the hydraulic approximation and it is assumed that the recovery factor is equal to unity; use is made of the Reynolds analogy. No general analysis of the singularities of the resulting ordinary differential equation is given. *J. Kestin.*

Guman, E. Heat-loss and pressure-drop in pipelines transporting heated oil. Acta Tech. Acad. Sci. Hungar. 17 (1957), 253–287. (German, French and Russian summaries)

Vlasov, A. A. On the transfer of mass and charge by means of surface waves. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **27** (1954), 224–242. (Russian)

Trlifaj, Miroslav. A theory of non-radiative transfer of excitation energy in solids. Czechoslovak J. Phys. **7** (1957), 1–10. (Russian summary)

Kapur, J. N. Uniqueness of maximum pressure for the cubic form-function in the general theory of composite charges. Proc. Indian Acad. Sci. Sect. A. **45** (1957), 177–183.

O'Sullivan, D. G. Diffusion and simultaneous chemical reactions. III. The degree of localization achieved in cytochemical staining procedures. Bull. Math. Biophys. **18** (1956), 199–203.

[For parts I–II see same Bull. **17** (1955), 141–153, 243–255.] The ratio of the mass of dye deposited in the enzyme site to the total mass of dye produced by the action of the enzyme in a given time has been modified by replacing the total mass of deposit by the mass that would be produced if the intermediate were completely converted into the deposited due to give the localization factor F. A theoretical approximation for F has been calculated which has the properties that $F=1$ for complete localization and approaches zero for very poor localization. This factor F is a function of a dimensionless parameter λ and two expressions for F are given; one suitable for use when $\lambda<1$ while the other is suitable for calculations when $\lambda>1$. *C. G. Maple* (Ames, Ia.).

Hansen, Robert S.; and Mai, Ursula H. Idealized models for adsorption from solution. I. van der Waals adsorption from regular solutions. J. Phys. Chem. **61** (1957), 573–577.

The theory of ideal van der Waals adsorption from regular solutions of equal sized molecules is developed, and explicit limiting forms for high and low concentrations of preferentially adsorbed components and for slightly soluble systems are given.

From the authors' summary.

Kaeppeler, H. J.; und Baumann, G. Über Systeme mit chemisch reagierenden Komponenten im Gleichgewicht. I. Die Berechnung der Gemischzusammensetzung. Astronaut. Acta 3 (1957), 28–46.

Auf Grund statistischer Betrachtungen über chemische Reaktionen im Gleichgewicht werden die Ableitung der Gleichgewichtskonstanten für solche Reaktionen im isotherm-isochoren und isotherm-isobaren Fall sowie Beziehungen zu deren Berechnung aus tabulierten Werten der thermodynamischen Funktionen von Einzelgasen dargestellt. *Aus der Zusammenfassung der Autoren.*

See also: Kim, p. 40; Crank, p. 66; Parrott, p. 85; Tanaev, p. 88; Jones, p. 102.

Quantum Mechanics

✱ Кузнецов, Б. Г. [Kuznecov, B. G.] Основы теории относительности и квантовой механики в их историческом развитии. [Foundations of the theory of relativity and quantum mechanics in their historical development.] Izdat. Akad. Nauk SSSR, Moscow, 1957. 328 pp. 13.50 rubles.

The main features of the present-day scientific picture of the world, namely the general ideas of relativity and quantum mechanics, are described in a form accessible to a wide circle of readers, engineers, teachers and students. The reader is expected to know the elements of the differential and integral calculus. Other parts of mathematics necessary for understanding the physical ideas are explained in elementary and non-rigorous fashion in the book itself. *Author's summary.*

Cap, F. Une interprétation causale de la théorie quantique est-elle possible? Ann. Inst. H. Poincaré **15** (1956), 113–122.

The author points out that the proof of the non-existence of hidden parameters (v. Neumann) is valid only for linear quantum theories. He discusses the possibility of describing some quantum effects with the help of nonlinear terms in non-quantized classical theories. Several formal examples are given (eigenvalues in H atom, oscillator, and systems of oscillators). *J. Plebański.*

Fano, U. Description of states in quantum mechanics by density matrix and operator techniques. Rev. Mod. Phys. **29** (1957), 74–93.

A very readable review of the uses of the density matrix in quantum mechanics. The emphasis is on the physical principles involved in its application to situations where less than maximal information is available about states.

A wide variety of examples is given ranging from the orientation of particles of arbitrary spin to irreversible processes. *A. S. Wightman* (Princeton, N.J.).

Heitler, W. Le principe du bilan détaillé. Ann. Inst. H. Poincaré **15** (1956), 67–80.

A mainly expository account of the principle of detailed balance, its generalizations and their application to prove the second law of thermodynamics. An example of Ehrenfest is first described in which detailed balance does not hold, but nevertheless a weakened form of it does. This latter form, first established by Stueckelberg [Helv. Phys. Acta **25** (1952), 577–580; MR **16**, 840], says that total transition probability into a state is equal to that out of it. The author gives a proof of it for the case of two colliding particles using the scattering matrix formalism. He applies it to give a derivation of the second law of thermodynamics. Finally he shows following Coester [Phys. Rev. (2) **84** (1951), 1259; MR **13**, 713] that the principle of microscopic reversibility may be used to prove a weakened form of the principle of detailed balance for pairs of colliding systems which says that the total transition probability from one state to another is the same as for the reverse process provided one averages over the spins of the systems. *A. S. Wightman.*

Shirokov, Iu. M.; and Sannikov, D. G. On the problem of unquantized relativistically invariant renormalized equations for a three-dimensional extended particle. Soviet Physics. JETP **4** (1957), 13–19.

A system of relativistically invariant equations for a smeared-out particle interacting with a field has been obtained by using the method previously proposed for constructing a three-dimensional extended particle. The conservation laws for the total energy-momentum four-vector are formulated. The particle is stable without the necessity for introducing additional forces (of the Poincare pressure type). An exact mass renormalization is carried out. For comparison with earlier equations, a rigorous limiting transition to the case of a point particle is made. Interaction with the electromagnetic field and with a scalar meson field are considered. *Authors' summary.*

Skyrme, T. H. R. Collective motion in quantum mechanics. Proc. Roy. Soc. London. Ser. A. **239** (1957), 399–412.

A method is described where the collective coordinates of a system appear not, as in most of the previous treatments, as redundant coordinates, but as additional coordinates needed to describe a "band", that is a set of levels phenomenologically associated with a collective mode. It is assumed that the energy eigenvalues E_n of these sets form a spectrum similar to that of a "model" Hamiltonian h operating in a space of coordinates ξ and possessing the eigenvalues e_n, $h\varphi_n(\xi)=e_n\varphi_n(\xi)$, and that the differences $E_n-e_n=e$ are approximately constant. A transformation function between $\varphi_n(\xi)$ and the eigenfunctions of the complete set $\psi_n(q)$ is then defined by $F(q, \xi)=\sum \psi_n(q)\varphi_n^*(\xi)$ which satisfies the equation

$$(H-h-e)F(q, \xi)=0.$$

Approximate solutions of this equation are found by a variational method. Suitable trial functions for F are suggested by its physical meaning. The method is applied to the anharmonic oscillator, plasma oscillations, to several nuclear models and to problems in meson field theory. *E. Gora* (Providence, R.I.).

Trainor, L. E. H. The formation of antisymmetric wave functions. Canad. J. Phys. **35** (1957), 555–561.

The problem of forming antisymmetric wave functions is discussed, and it is shown that a convenient method developed previously by the author [Phys. Rev. (2) **85** (1952), 962–972; **95** (1954), 801–810] leads to the usual results. *Author's summary.*

Borgardt, A. Pseudoscalar interaction in non-linear mesodynamics. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 42–43. (Russian)

A somewhat general form of Lagrangian is chosen so as to lead to non-linear field equations. The possible specific forms of the Lagrangian are determined by imposing certain conditions. The corresponding static field equations are given in the limit of a negligible mass-term, and some of their solutions are discussed. *N. Rosen* (Haifa).

Jouvet, B. Fermi coupling and mass and charge spectra of bosons. Nuovo Cimento (10) **5** (1957), 1–20.

The author here continues his efforts [Nuovo Cimento (10) **2** (1955), supplemento, 941–968; MR **17**, 927] to make a realistic theory of elementary particles starting from Fermi interactions between spin $\frac{1}{2}$ particles. He is led to use and interpret the formalism of field theory in very unconventional ways. For example, in the usual theory when a cut off is introduced, the fields satisfy the ordinary canonical commutation relations. Only in the limit of infinite cut off do the commutators have worse than delta function singularities. In the author's theory, the latter case prevails with or without cut off. By such strong measures, whose internal consistency the reviewer finds it difficult to judge, he shows that a theory of a spinor field interacting with itself via a scalar Fermi interaction is equivalent to the theory of the same spinor field interacting with a scalar field via a scalar Yakawa interaction, the scalar field describing unstable spin zero mesons. The known renormalizability of the theory then is shown to imply the renormalizability of the Fermi theory. It is fundamental for the program that certain equations for renormalization constants be regarded not as identities but as equations to be solved for the masses and coupling constants of the particles. The latter half of the paper criticizes various arguments that the above mentioned equations are indeed identities. Here the main problem is to show that Källén's proof [Danske Vid. Selsk. Mat. Fys. Medd. **27** (1953), no. 12; MR **15**, 79] of the infinity of at least one of the renormalization constants is incorrect. In the opinion of the reviewer the author's criticisms fails to shake the proof. His main point is that Källén uses a divergent series of positive terms to represent a function and then accepts one term as a minorant. Actually the series is convergent since it is essentially of the form $\|\Phi\|^2=\sum_i |(\Phi, \Phi_i)|^2$, where Φ_i is an orthonormal set. *A. S. Wightman* (Princeton, N.J.).

Sudakov, V. V. Meson-meson scattering in quantized meson field theory. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 338–340. (Russian)

Dyatlov and Ter-Martirosyan have derived an analogue for meson-meson scattering of the "ladder" approximation Beth-Salpeter equation, and have determined the asymptotic form of its solution for certain regions of large momenta [Ž. Eksper. Teoret. Fiz. **30** (1956), 416–419]. In general, the special case in which the four momentum of all initial and final mesons is the same could only be treated by numerical integration of their equations. In the

present note the author derives an equation for this special case directly and uses it to determine the asymptotic behavior of the scattering amplitude. The result is in agreement with that of Dyatlov and Ter-Martirosyan where comparison is possible. It justifies an approximation made by Pomeranchuk in his attempt to prove the inconsistency of meson theory [Dokl. Akad. Nauk .SSSR (N.S.) **103** (1955), 1005–1008; MR **17**, 565].

A. S. Wightman (Princeton, N.J.).

Lomsadze, Yu. M. On the potential in pair theories of nuclear forces. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 545–548. (Russian)

The pair theory of nuclear forces in which nucleon and meson are assumed to have spin one half and to be coupled by a Fermi interaction yields very singular nucleon potentials. The author considers the question whether the singularity will be reduced as a result of renormalization and a full relativistic treatment. He assumes a pseudo scalar Fermi interaction and adds to his Lagrangian density two counter terms of the form

$$\delta\kappa[\bar{\psi}(x)\gamma_5\psi(x)]^2 \text{ and } k\left[\frac{\partial}{\partial x^n}(\bar{\psi}(x)\gamma_5\psi(x))\right]^2.$$

Here, ψ is the spinor field of the nucleon, and $\delta\kappa$ and k are renormalization constants chosen so as to make the scattering amplitude of two nucleons finite in the second order of perturbation theory. In that approximation the potential turns out to be somewhat more singular then in the unrenormalized theory. The author concludes that in so far as perturbation theory is justified the pair theory yields too singular potentials to be satisfactory. He mentions that similar results hold for other variants of the Fermi interaction and also for non-linear meson interactions whose interaction Lagrangian density is of the form $f\bar{\psi}(x)\psi(x)\phi(x)^*\phi(x)$, where ϕ is a scalar or pseudoscalar field.

A. S. Wightman (Princeton, N.J.).

Schmidt, W.; und Baumann, K. Quantentheorie der Felder als Distributionstheorie. Nuovo Cimento (10) 4 (1956), 860–886.

The authors seek a set of axioms to characterize the notion of a local field in relativistic quantum field theory. (Previous efforts along these lines [e.g., K. O. Friedrichs Mathematical aspects of the quantum theory of fields, Interscience, New York, 1953; MR **15**, 80] were not completely successful in isolating the basic ideas which occur in the physical literature of the last thirty years.) Such an axiomatization is implicitly contained in a number of recent papers on technical questions of field theory [see, e.g., R. Haag, Mat. Fys. Medd. Dan. Vid. Selsk. **29** (1955), no. 12; MR **17**, 112; H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento (10) **1** (1955), 205–225; MR **17**, 219; A. S. Wightman, Phys. Rev. (2) **101** (1956), 860–866; MR **18**, 781], but the present authors are the first to bring it into the open. Their axioms (when slightly amended as indicated below) define a physically reasonable and mathematically elegant notion of a local field which deserves the full scrutiny of mathematicians. The axioms are in four parts. First, assumptions common to most versions of relativistic quantum theory: The existence of a separable Hilbert space, $\mathfrak{H}$, whose vectors represent states, and in $\mathfrak{H}$, a representation up to a factor of the inhomogeneous Lorentz group, $\{a, \Lambda\}\to U(a, \Lambda)$; the absence of negative energy states in $\mathfrak{H}$ and the existence of a vacuum state, Ψ_0, invariant under the $U(a, \Lambda)$. Second, the axioms for a field (for simplicity, only the case of a neutral scalar field will be given here): Let $\mathscr{D}$ be the set of complex valued infinitely differentiable functions defined on space time and vanishing "rapidly" at infinity. Then a field, ϕ, is an hermitean-operator valued distribution i.e. a correspondence $f\to\phi(f)$ which gives for each $f\in\mathscr{D}$ a linear operator in $\mathfrak{H}$ with the properties:

$$(\mathrm{I}_2)\quad \phi(\alpha f)=\alpha\phi(f),\ \phi(f_1+f_2)=\phi(f_1)+\phi(f_2),\ \phi(f)^*=\phi(\bar{f}).$$

(Precisely what vanishing "rapidly" should mean is not quite clear physically. The authors choose $\mathscr{D}$ as the functions vanishing faster than any inverse power of the distance. It is probably physically more natural to choose the functions vanishing outside compact sets.) (I_1) $\phi(f)$ is weakly continuous in f, i.e. the scalar products $(\Phi, \phi(f)\Psi)$ where $\Phi, \Psi\in\mathfrak{H}$ are distributions in the sense of Schwartz. (Unfortunately, at this stage the authors fail to make clear a point of some mathematical and physical importance. They apparently implicitly require that $\phi(f)$ should be bounded. In the physically interesting cases $\phi(f)$ is unbounded, therefore (I_2) and (I_3) have to be supplemented by some such axiom as the following. (I) The $\phi(f)$, $f\in\mathscr{D}$ possess a common dense domain D such that $\Psi_0\in D$, $\phi(f)D\subset D$ and $U(a, \Lambda)D\subset D$. All equations of the other axioms then are assumed to hold on D. The authors actually assume strong continuity of the $\phi(f)$ in f which is not as physically natural as (I_1). Third, the transformation law of the field (II) $U(a, \Lambda)\phi(f)U(a, \Lambda)^{-1}=\phi(f')$, where $f'(x)=f(\Lambda^{-1}(x-a))$ for Λ not inverting time and $f'(x)=\bar{f}(\Lambda^{-1}(x-a))$ otherwise. Fourth, local commutativity of the field: (III) $[\phi(f), \phi(g)]=0$ for all pairs $f, g\in\mathscr{D}$ such that $f(x)g(y)=0$ whenever $x-y$ is a time-like or light-like vector. Starting from these axioms, the authors repeat calculations made by the reviewer in the paper referred to above which show that the $\phi(f)$ can be realized in a certain Hilbert space constructed from the vacuum expectation values $(\Psi_0, \phi(f_1)\cdots\phi(f_m)\Psi_0)$. The proof as given is not complete because it does not demonstrate the joint continuity of this vacuum expectation value in its n arguments, but that oversight can easily be repaired with the help of the nuclear theorem of L. Schwartz [Proc. Internat. Congress Math. Cambridge, Mass., 1950, v. I, Amer. Math. Soc., Providence, R.I., 1952, pp. 220–230; MR **13**, 562]. They then display this realization explicitly for three cases: a free neutral scalar field, a free spinor field, and the electromagnetic field. The paper also contains a short review of the terminology of distribution theory, a definition of the notion of a distribution whose arguments are implicitly defined functions, and a description of the main Lorentz invariant distributions satisfying the wave equations, $(\Box+m^2)f=0$ or $-\delta$.

A. S. Wightman (Princeton, N.J.).

Solov'ev, V. G. The propagation function of a nucleon in quadratic approximation. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 578–581. (Russian)

The author states the results of an investigation of the Green's functions of pseudo-scalar meson theory in an approximation different from that of perturbation theory, one which is naturally suggested by the representation of Green's functions in terms of functional integrals. The approximation is obtained by dropping certain terms in the functional equation for the nucleon Green's function in an external meson field. Then the simplified equation is solved exactly by the method of Fok [Phys. Z. Sowjetunion **12** (1937), 404–425]. Using the solution, an integral representation of the one-nucleon Green's function is found in terms of the Fredholm determinant of a certain

Fredholm integral equation. To calculate other Green's functions it was found necessary to develop analogous approximation methods for the variational derivatives with respect to the external meson field of the nucleon Green's function in the external field. Results are given for the vertex parts corresponding to the interaction of nucleon with meson and electromagnetic field. They are expressed in terms of the solution and Fredholm determinant of a certain Fredholm equation. The result leads to a neutron and proton magnetic moment satisfying $\mu_p+\mu_n=1$. *A. S. Wightman.*

Gell-Mann, Murray; and Brueckner, Keith A. Correlation energy of an electron gas at high density. Phys. Rev. (2) **106** (1957), 364–368.

The ground state energy of an electron gas (with neutralising positive background) consist of (i) the Fermi energy, (ii) the exchange energy due to first order Coulomb interaction, (iii) correlation energy, being the higher order terms. If the volume per electron is $\frac{4}{3}\pi r_s^3$ (r_s measured in Bohr radii), then the correlation energy is known to be $A \ln r_s+C+O(r_s)$, where A is 0.0622. The computation of the higher terms by straight-forward perturbation theory is obviated by the appearance of divergent integrals. Nevertheless the authors succeed in giving an exact expression for C by selecting from the divergent terms those contributions that do not vanish for $r_s\to 0$. The method employs Feynman diagrams and formal time variables in analogy with field theory techniques. After summing these contributions of all orders, the integrals are finite, and the result is $C=-0.096$. *N. G. van Kampen.* (Utrecht).

Gell-Mann, Murray. Specific heat of a degenerate electron gas at high density. Phys. Rev. (2) **106** (1957), 369–372.

The method of the article reviewed above is used to compute the energy of states with one excited electron. The expansion in powers of r_s again contains divergent integrals. It is again possible to obtain a finite integral by selecting in each term the most divergent contribution and summing these contributions before integrating. This leads to the following expression for the ratio of specific heats at zero temperature of a Fermi gas with and without Coulomb interactions:

$$c_{\text{coul}}/c_0=[1+0.083r_s(-\ln r_s-0.203)+\cdots]^{-1}.$$

(The numerical factors stand for exactly defined quantities.) *N. G. van Kampen* (Utrecht).

Sawada, Katuro. Correlation energy of an electron gas at high density. Phys. Rev. (2) **106** (1957), 372–383.

The problem of Gell-Mann and Brueckner (see above) is approached in a different way, avoiding perturbation theory and manipulations with divergent integrals. First the Hamiltonian is written in terms of creation and annihilation operators. These are arranged in such a way that the diagrams retained by Gell-Mann and Brueckner appear as separate terms; the other interaction terms are omitted. The remaining problem is similar to the problem of a heavy particle interacting with a neutral scalar field, which has been solved exactly. Indeed, it is again possible to obtain an exact equation for the eigenvalues and, in addition, to find the value of the Coulomb energy in the new ground state. The result agrees numerically with that of Gell-Mann and Brueckner. It is also checked that a suitable expansion of the present result leads to the perturbation series. *N. G. van Kampen* (Utrecht).

Pauli, W. Remarks on problems connected with the renormalization of quantized fields. Nuovo Cimento (10) **4** (1956), supplemento, 703–710.

Paper read at the Pisa Conference on Elementary Particles in 1955. Pauli had shown for the simplified model of meson theory given by Lee [Phys. Rev. (2) **95** (1954), 1329–1334; MR **16**, 317] that the renormalization of the coupling constant gives rise to an indefinite metric ('ghost states' with negative probabilities). Similarly, in a simplified model for a charged particle in interaction with the electromagnetic field, renormalization gives rise to an indefinite Hamiltonian ('self-accelerating solutions' with negative energies). The bearing of these facts on the state of field-theory is discussed.
N. G. van Kampen (Utrecht).

Edwards, S. F. Recent Birmingham work on the solution of quantum field theory. Nuovo Cimento (10) **4** (1956), supplemento, 711–722.

Paper read at the Pisa Conference on Elementary Particles in 1955. The solution of the equations of quantum field theory can formally be written in terms of 'Green functions'. These can be written by means of functional integrations with respect to the fields. The aim is to evaluate these integrals without recourse to the usual perturbation expansion. The difficulties encountered (in the case of a nucleon in interaction with pseudo-scalar mesons) are mentioned, and attempts to overcome them are discussed. *N. G. van Kampen* (Utrecht).

Mitra, A. N. Multiple π^0-production in anti-proton annihilation. Nuclear Phys. **1** (1956), 571–580.

The author calculates the cross section for the multiple production of an arbitrary number of π^0 mesons in proton-antiproton annihilation by closely following the treatment which has been used by the reviewer in the investigation of multiple photon production in quantum electrodynamics [Phys. Rev. (2) **98** (1955), 1502–1511; MR **17**, 442]. However, he makes use of some approximations which are justified in quantum electrodynamics but are not justified in mesonic interactions. *S. N. Gupta.*

Umezawa, H.; and Visconti, A. Commutation relations and relativistic wave equations. Nuclear Phys. **1** (1956), 348–354.

This paper describes a general method of obtaining the commutation relations for fields of arbitrary spin in the absence of interaction. In the special case of spin 3/2, the general result is in agreement with that obtained earlier by the reviewer [Phys. Rev. (2) **95** (1954), 1334–1341; MR **16**, 321]. *S. N. Gupta* (Detroit, Mich.).

Schöpf, Hans-Georg. Erhaltung und Invarianz. Ann. Physik (6) **18** (1956), 278–287.

A well-written discussion of the fact that invariance properties of the Lagrangian imply various conservation theorems. No reference is made to the many previous treatments of the same theme such as that of Hill [Rev. Mod. Phys. **23** (1951), 253–260; MR **13**, 503].
A. J. Coleman (Toronto, Ont.).

Feĭnberg, E. L.; and Černavskiĭ, D. S. Higher approximations in the method of self-consistent field in the meson theory. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 619–622. (Russian)

Using the method of separation of self-energy proposed earlier [same Dokl. (N.S.) **103** (1955), 421–424], the authors

investigate the higher approximations in the calculation of the interaction between nucleons. They conclude that, if the interaction energy operators for different quantum momenta commute, the higher approximations are important; otherwise the one-meson approximation is sufficient to give a satisfactory description. *N. Rosen.*

Ovsyannikov, L. V. A general solution of the renormalization group equations. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 1112–1114. (Russian)

A general solution is found of the functional equations of the renormalization group for the Green's function of quantum electrodynamics as obtained through the use of perturbation theory by N. N. Bogolubov and D. V. Shirkov [same Dokl. (N.S.) **103** (1955), 203–206; MR **17**, 441]. Similarly the solution of the corresponding equations in meson theory [D. V. Shirkov, ibid. **105** (1955), 972–975; MR **17**, 1033] is obtained. *N. Rosen* (Haifa).

Polkinghorne, J. C. General dispersion relations. Nuovo Cimento (10) **4** (1956), 216-230.

A formalism is developed whereby dispersion relations may be obtained for all meson-nucleon scattering processes, including those in which mesons are created or destroyed. A suitable kinematical description of the processes is first given and a new amplitude is defined in terms of it. This amplitude has two important properties. Firstly it coincides with the Feynman amplitude for positive energies of the incoming and outgoing particles. Secondly it satisfies a certain causality condition. These enable one to obtain dispersion relations. The significance of these results in affording a new way of formulating quantum field theory is discussed. (Author's summary.) *N. Rosen* (Haifa).

Logunov, A. A. The spectral representation and the renormalization group. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 740–742. (Russian)

From the functional equations holding for the photon and electron Green's functions in quantum electrodynamics relations are obtained for the spectral representations of these functions. Asymptotic expressions are obtained for the latter under certain restrictions. *N. Rosen* (Haifa).

Królikowski, W.; and Rzewuski, J. Relativistic two-body problem in one-time formulation separation of angular variables in the case of one-quantum interaction in electrodynamics. Acta Phys. Polon. **15** (1956), 321–341 (1957). (Russian summary)

This is the part II of a paper by the same authors [Nuovo Cimento (10) **2** (1955), 203–219; MR **17**, 334] in which the two-fermion problem in quantum field theory was transformed into the one-time equation possessing the form of a Schrödinger equation. In this paper the authors proceed with their calculations. They succeeded in solving the angular part of this problem and use the solution for the separation of angular variables. The resulting set of integral equations in one variable is then investigated. The solution of this equation seems very difficult "if at all possible". *L. Infeld* (Warsaw).

Umezawa, H.; and Visconti, A. Renormalisation and mass levels. Nuclear Phys. **1** (1956), 20–32.

A spinor particle with a finite number of stable states of different masses is considered using the theory of propagators [Umezawa and Visconti, Nuovo Cimento (10) **1** (1955), 1079–1103; MR **17**, 443]. Corresponding to each mass value $m^{(j)}$ a constant $Z^{(j)}$ is defined which is the normalisation factor of the propagator of the particle whose mass is $m^{(j)}$ and is an extension of Dyson's Z_2 to many mass levels. It then appears that half of the $Z^{(j)}$ are positive, half negative. This implies that the interaction Hamiltonian is not hermitian and the S-matrix not unitary. The expectation value of the number operator for the state of mass $m^{(j)}$ is negative for $Z^{(j)}<0$. This situation may be avoided by giving a special distribution of singularities to the function $a(x)$ where

$$-i\gamma_\mu p_\mu-\kappa-\mathfrak{M}(-i\gamma p)=a(-i\gamma p)\prod_{j=1}^{n}(-i\gamma_\mu p_\mu-m^{(j)}).$$

Lee's model [Phys. Rev. (2) **95** (1954), 1329–1334; MR **16**, 317] is discussed and also the renormalisation scheme of quantum electrodynamics is briefly reconsidered with particular reference to the renormalisation factor Z_3 of the photon propagator. *C. Strachan* (Aberdeen).

Brueckner, K. A.; and Wada, W. Nuclear saturation and two-body forces: self-consistent solutions and the effects of the exclusion principle. Phys. Rev. (2) **103** (1956), 1008–1016.

Brueckner and his collaborators have developed an approximation method for the treatment of many-body problems [Brueckner, Phys. Rev. (2) **96** (1954), 508–516; **97** (1955), 1353–1366; **100** (1955), 36–45; Brueckner and Levinson, ibid. **97** (1955), 1344–1352; Brueckner, Levinson, and Mahmoud, ibid. **95** (1954), 217–228; Brueckner, Eden, and Francis, ibid. **98** (1955), 1445–1455]. This method is based on a combination of the self-consistent field method and the quantum theory of multiple scattering developed by K. M. Watson [ibid. **89** (1953), 575–587; MR **14**, 829], and leads to an approximate reduction of the exact equation for the many-body reaction matrix to a corresponding equation for a two-body reaction matrix. This equation differs from the equation for the reaction matrix in scattering theory in two respects. First, the energy differences include the interaction energy of a particle with the rest of the system; this is taken into consideration by the introduction of an "effective mass". Second, transitions to occupied states are forbidden by the exclusion principle. Both the effective mass approximation and the corrections for the exclusion principle are developed in this paper in more detail than in previous work. To illustrate the method a square-well potential is used. Procedures for obtaining approximate solutions and for numerical computations are outlined. Finally, an approximation method for taking into consideration the finite size of the nucleus is briefly discussed. *E. Gora* (Providence, R.I.).

Ascoli, R.; und Heisenberg, W. Zur Quantentheorie nichtlinearer Wellengleichungen. IV. Elektrodynamik. Z. Naturf. **12a** (1957), 177–187.

This paper is one of a long series [Heisenberg, MR **15**, 914, 915; **17**, 1031; **18**, 174; Heisenberg, F. Kortel and H. Mitter, MR **17**, 330] where Heisenberg and collaborators investigate a spinor field $\psi(x)$ coupled to itself according to $\gamma\,\partial\psi(x)/\partial x+l^2\psi(x)(\psi^+(x)\psi(x))=0$. The particular subject under discussion here is the appearance of long range forces between the spinor particles. The authors point out that no forces of this kind appear in a straight forward application of perturbation theory, but that the use of a modified singular function and a particular scheme of approximation produces such forces. If the

authors' computations are taken for granted, it appears that one also gets long range, strongly spin dependent forces. Whenever the authors average over spins in the initial and final states, these effects drop out and are not considered any further. The remaining terms have a form analogous to the electromagnetic interaction between two spin $\frac{1}{2}$ particles with a fine structure constant that depends on the details of the theory. For the particular model considered here, this constant is computed approximately and found to be 1/267. A special paragraph is devoted to an explanation of why a certain method of approximation which was used in the earlier papers is useless. The need for this explanation appears to be due to a few mistakes in sign found in the earlier papers as pointed out by Kita [Progr. Theoret. Phys. **15** (1956), 83–85]. When these mistakes are corrected some of the results of the earlier papers are changed. In particular, one does not obtain the special asymptotic form of the vacuum expectation value of the time ordered product of two fields that the authors desired. Finally, the inclusion of an isotopic spin in the model is discussed.
G. Källén (Copenhagen).

Gurzhi, R. I. On the scattering of photons by nucleons. Soviet Physics. JETP **3** (1957), 941–945.

The pion cloud around a nucleon produces deviations from the Klein-Nishina formula when we consider the scattering of photons on nucleons. This effect is considered in the present paper, using a semiphenomenological model of isobaric states. Such calculation was also carried out by Minami [Progr. Theoret. Phys. **9** (1953), 108–116], but he included absorption only at resonance. The present calculation is closely related to a calculation by Ritus [Ž. Eksper. Teoret. Fiz. **27** (1954), 660–676] which was carried out simultaneously but independently. The paper gives the main steps of the somewhat laborious calculation and gives a few numerical results for the total cross section. For example, at resonance (that is, at 340 Mev laboratory photon energy), the total cross section turns out to be 2.13 microbarns. *M. J. Moravcsik.*

Ginzburg, V. L. On relativistic wave equations with a mass spectrum. Acta. Phys. Polon. **15** (1956), 163–175.

The question of relativistic wave equations having a mass spectrum and containing new continuous variables is discussed. Mostly the case for which the new variables are components of some space-like four-vector is considered, since similar equations have been very often used in recent times in connection with attempts to build a nonlocal field theory. *Author's summary.*

Yafet, Y. The g value in conduction electron spin resonance. Phys. Rev. (2) **106** (1957), 679–684.

The departure of the g factor from the free-spin value due to spin-orbit interaction is calculated for conduction electrons in a nondegenerate band. The Kohn-Luttinger representation is used and the result is obtained to second order in the wave-vector of the electron at the Fermi surface. The result is expressed in two forms, one of these being a sum of volume and surface integrals over the unit cell. The orders of magnitude of these are discussed but no numerical calculation are made. *Author's summary.*

Davison, B. Spherical-harmonics method for neutron-transport problems in cylindrical geometry. Canad. J. Phys. **35** (1957), 576–593.

"The paper deals with the application of the spherical-harmonics method to systems with complete cylindrical symmetry, that is, to systems invariant under rotation around and translation parallel to an axis. Closed-form expressions are given in an arbitrary order of approximation both for the spherical-harmonics moments in terms of the constants of integration and, conversely, for the constants of integration in terms of the spherical-harmonics moments. This removes the need for numerical inversion of matrices and simplifies the treatment of multilayer problems." (Author's summary.) *T. E. Hull.*

Baier, V. N.; and Pekar, S. I. Nucleomesodynamics in strong coupling. II. The ground and isobar states, nucleon charge and spin. Soviet Physics. JETP **3** (1956), 340–350.

English translation of Ž. Eksper. Teoret. Fiz. **30** (1956), 317–329 [MR **18**, 174.].

Szépfalusy, P. Die Hartree-Focksche Methode im Falle eines nichtorthogonalen Einelektronwellenfunktionen-Systems. Acta Phys. Acad. Sci. Hungar. **6** (1956), 273–292. (Russian summary)

The ordinary Hartree-Fock method can be summarized in the following way. To describe an atom with n electrons, we take an otrhonormal set of functions $\psi_n^0(q)$ and form the determinant $\phi(q_1, \cdots, q_n)=\mathrm{Det}\ \|\psi_i^0(q_j)\|$. The best possible wave function ϕ of this kind is obtained when the total energy of the atom is as small as possible, or when the functions ψ^0 fulfill the "self consistency" requirement $H^{F_0}\psi^0(q)=E^0\psi^0(q)$ with $H^{F_0}=H_0+U^0+A^0$. Here, H_0 is the sum of the kinetic energy of an electron and the electrostatic energy of the electron in the field of the nucleus, U^0 is the Coulomb energy of the electron in the average field of all the other electrons, and A^0 is the corresponding exchange term (A=Austausch=exchange in German). The author shows how to modify this procedure in such a way that we can use a set of functions $\psi_n(q)$ which are not necessarily orthonormal. He does this by writing $\psi(q)=\psi^0(q)C^0$ with a non-unitary matrix C^0 and by substituting this into the formalism above. His final relation reads

$$-\frac{\hbar^2}{2m}\Delta r\psi_i(q)+\frac{1}{2m}(p_{r_i}{}^{0*}-p_{r_i}{}^2)\psi_i(q)+\frac{\hbar}{2m}\frac{l(l+1)}{r_i{}^2}\psi_i(q)$$
$$-\frac{Ze^2}{r}\psi_i(q)+U\psi_i(q)-A\psi_i(q)=E_i\psi_i(q).$$

Here, Δ_r is the radial part of the Laplacian, U and A have a meaning analogous to U^0 above, and the symbol $(p_r{}^0)^2$ is defined by $-\hbar^2\Delta_r\psi_i{}^0(q)=(p_{r_i}{}^0)^2\psi_i{}^0(q)$, while $(p_r{}^2 2m)\psi_i(q)$ is the same as the first term in the equation above. It must be remarked that this is not an equation for $\psi(q)$ alone, since the orthonormal functions $\psi^0(q)$ still enter into it. Furthermore, it appears somewhat doubtful to the reviewer whether it is possible in general, to obtain better approximations to the charge distribution or energy levels of the atom by using the functions $\psi(q)$ instead of the functions $\psi^0(q)$, since the former can be obtained from the latter with the aid of the matrix C^0 above. However, for the special applications that the author has in mind, these last remarks of the reviewer do not seem to be very relevant (see the paper reviewed below). *G. Källén.*

Szépfalusy, P. Die Inhomogenitätskorrektion der Fermischen kinetischen Energie von Teilchen mit halbzahligem Spin. Acta Phys. Acad. Sci. Hungar. **6** (1956), 293–305. (Russian summary)

With the aid of his technique developed in the paper

reviewed above the author studies the total energy of an atom in the statistical model of Thomas-Fermi-Dirac. Using semiclassical considerations, the second term in the equation displayed in the preceding review is interpreted as the difference between the radial kinetic energy of the electron and a zero point kinetic energy. The first term is transformed into an integral over the square of the gradient of the electron density. Expressions of this kind were first used by Weiszäcker [Z. Physik **96** (1935), 431–458] in the statistical theory of the atom. The author shows that the expression he obtains in this way is identical with what has previously been obtained by Gombás [Acta Phys. Acad. Sci. Hungar. **3** (1953), 127–154; **5** (1956). 483–502]. *G. Källén* (Copenhagen).

Sokolow, A. A. On the possibility of the excitation of macroscopical oscillations by the quantum fluctuations ("macroatom"). Bul. Inst. Politehn. Iaşi (N.S.) **2** (1956), 43–49. (Romanian summary)

This translation into English of same Bul. (N.S.) **2** (1956), 39–42 [MR **18**, 542] was made by the editorial staff of the Bul. Inst. Politehn. Iaşi.

Moshinsky, Marcos. Collective motions and nuclear reactions. Rev. Mexicana Fis. **5** (1956), 1–41. (Spanish. English summary)

The collective motions are described in terms of surface deformation and coupling as proposed by A. Bohr [Danske Vid. Selsk. Mat.-Fys. Medd. **26** (1952), no. 14]. The inelastic scattering of nucleons is treated by a perturbation method. The validity of the method is discussed, and the latter is found to fail at certain resonance energies. A Tamm-Dancoff method, in which a definite number of states is considered, is introduced, enabling a rigorous analysis to be carried out. The results are in general agreement with those of the cloudy crystal ball model. The case of strongly deformed target nuclei is investigated. The description of collective effects in nuclear reactions is discussed in analogy with the dispersion of sound waves by soap bubbles. *N. Rosen.*

Muhtarov, A. I.; and Černogorova, V. A. Photoproduction of neutral mesons. Akad. Nauk Azerbaĭdžan. SSR. Dokl. **12** (1956), 77–80. (Russian. Azerbaijani summary)

Results are presented of calculations based on the operator division method of Sokolov [A. Sokolov and D. Ivanenko, The quantum theory of fields, Gostehizdat, Moscow-Leningrad, 1952; MR **14**, 1044]. Expressions are given for the cross-sections for photoproduction of neutral mesons for various spin states of the nucleons, for the cases of scalar and pseudoscalar mesons. Curves are given for the angular dependence, and comparison is made with experiment. *N. Rosen* (Haifa).

★ **Hartree, D. R. The variation of atomic wave functions with atomic number.** International conference on current problems in crystal physics. pp. 47–65. Massachusetts Institute of Technology, Cambridge, Mass., July 1–5, 1957.

A systematic survey of the variation of atomic wave functions with respect to atomic number was undertaken to provide a means of interpolating wave functions as accurately as possible. The wave functions are scaled with respect to a characteristic length, R, to give a "reduced wave function". For an atom of atomic number N, R is related to the effective nuclear charge, $N-\sigma$, σ being the screening parameter. The value of σ depends on the characteristic length which is used; several are suggested but the mean radius was found to be most satisfactory.

For large N the wave functions are represented asymptotically by a series in $1/N$. From it the limiting values as $N\to\infty$ of σ and the reduced wave functions are determined, which make the estimation of wave functions for atoms of atomic number higher than any for which results are available, one of interpolation instead of extrapolation.

Plots are given of the screening parameter and the reduced wave functions as functions of the mean radius; the relationship is very nearly linear and therefore suitable for interpolation. This method of determining screening parameters is compared with those used in calculating atomic scattering factors. *C. Froese.*

Vasilache, Sergiu. Mathematical problems in the domain of nuclear energy. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat. Fiz. (3) **10** (1956), no. 2(17), 29–54. (Romanian)

This note is a theoretical study of nuclear reactors, in the center of which stands the integro-differential transport equation. A short summary of the contents of the note: An introduction in which it is emphasized that a researcher in the domain of nuclear energy must be acquainted with all the branches of modern mathematics, a paragraph about the concepts of nuclear physics; a paragraph about the theoretical study of nuclear reactors in which the transport equation is deduced; a paragraph about some particular cases in which among others an infinite medium with a spherical cavitation and a point source in the center of the cavitation is considered; a paragraph about the exact solution of the transport equation and a paragraph about the particular study of the diffusion of thermal monochromatic neutrons.

B. Germansky (Jerusalem).

Strachan, Charles. Formation of an alpha-group in the shell-model of a heavy nucleus. Proc. Cambridge Philos. Soc. **53** (1957), 494–507.

By means of a variation principle applied to a nuclear wave function for four nucleons in the field of a closed core, equations are derived giving the relative proportions of a component describing an alpha-particle group and components describing the four nucleons in shell-model states with spin-orbit coupling wave functions, the two neutron states and the two proton states each paired to give zero resultant angular momentum. Numerical calculations, taking into account only $1h_{9/2}$ states for protons and the $2g_{9/2}$ states for neutrons, give the alpha-group wave function a relative weight (in the linear combination) of about 1%. (Author's summary.)

F. Rohrlich (Iowa City, Iowa.).

Samoilovich, A. G.; and Kondratenko, V. M. On the theory of atomic semiconductors. Soviet Physics. JETP **4** (1957), 481–491.

The authors extend the model of semiconductors developed by Shubin and Vonsovskii [Phys. Z. Sowjetunion **7** (1935), 292–328]. In this model the atoms form an ideal crystal lattice, each atom having a single bound s-electron. Conductivity is brought about by the formation of holes and doubly-occupied sites. The treatment is entirely formal, most of the paper being taken up with the mechanics of the expression of the hamiltonian by means of creation and annihilation operators which are used to

describe the electronic transitions. The formal expressions are carried out to third-order terms, with allowance for excitation of the electrons at each site. The theory is applied to a qualitative discussion of the kinetics of photoconductivity. *E. L. Hill* (Minneapolis, Minn.).

Kalos, M. H.; Biedenharn, L. C.; and Blatt, J. M. Numerical calculations for the neutron-proton system with tensor forces. Nuclear Phys. **1** (1956), 233–244.

Low-energy properties of the neutron-proton system are calculated for various combinations of central and tensor force. The potentials considered are square well, exponential, Yukawa, Gauss and Morse with variable ranges and strengths. From these are computed numerically the quadrupole moment, scattering lengths and effective ranges, the amount of D-state, and the shape dependent parameter in triplet state. *N. G. van Kampen.*

March, N. H. A variational method for the calculation of particle densities and sums of eigenvalues in wave mechanics. Proc. Phys. Soc. Sect. A. **70** (1957), 169–175.

This paper extends a suggestion of Macke and applies it to one-dimensional problems. The method is a variational one based on trial functions which are arbitrarily transformed plane waves; and he shows that it represents an improvement on both the Thomas-Fermi and von Weizsäcker methods. *P. W. Anderson.*

Frahn, W. E. Nucleon-nucleus interaction from the statistical model. Nuovo Cimento (10) **5** (1957), 393–401.

The statistical nuclear model with exchange forces leads to a non-local interaction between a nucleon and its neighbours in nuclear matter (Van Vleck-potential). The implications of this interaction with regard to the modification of nuclear motion and to the energy dependence of the real parts of the optical model parameters are examined. Finally, a derivation of Van Vleck's potential is given from Brueckner's coherent model.

Author's summary.

Úlehla, Ivan. Single-nucleon model of the nucleus. I. Czechoslovak J. Phys. **7** (1957), 11–19. (Russian-English summary)

Carrassi, M. The spin kinematics for a charged particle in a uniform magnetic field. Nuovo Cimento (10) **5** (1957), 955–960.

Tolhoek and De Groot showed [Physica **17** (1951), 17–32] among other things that an electron moving in a transversal, homogeneous magnetic field does not change the direction of its spin relative to its momentum during the motion. The original calculation was made only to first order in the magnetic field. In the paper reviewed here it is shown that this result is, in fact, exact in the external field for a spin $\frac{1}{2}$ particle with no anomalous magnetic moment. The proof consists of an explicit calculation using the well known solution of the Dirac equation with a constant magnetic field. The direction of the spin of a moving electron is defined with the aid of the polar angles x and ω from $B/A=\tan(\frac{1}{2}x)\exp(i\omega)$, where B/A is the ratio of the two "big" components of the wave function. *G. Källén* (Copenhagen).

Glover, Francis N.; and Chraplyvy, Zeno V. Reduction of relativistic wave equations and the "contact interaction". Phys. Rev. (2) **103** (1956), 821–824.

The large component method of reducing the Dirac one electron and the Breit two electron equations is exhibited in a form which permits comparison with the Foldy-Wouthuysen method. Sufficient conditions are obtained for the equivalence of the two methods to order c^{-4}. Contrary to a contention of Wu and Tauber [Phys. Rev. (2) **100** (1955), 1767–1770] the large component method gives rise to the contact interaction term. In the appendix the integral of $\psi^*\{\mathbf{A}\cdot\text{curl}(\mathbf{B}\times\mathbf{M})/r^3\}\psi$ is evaluated. *A. J. Coleman* (Toronto, Ont.).

Królikowski, W.; and Rzewuski, J. One-time formulation of the relativistic two-body problem. Separation of angular variables. Nuovo Cimento (10) **4** (1956), 975–990.

The one-time equation for the relativistic two fermion problem obtained in previous papers of the authors [Nuovo Cimento (10) **2** (1955), 203–219; Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 353–354; MR **17**, 334, 441] is written out explicitly with kernels corresponding to 1) single particle theory of fermions, and 2) symmetrical hole theory considering one-particle interaction only. In the present paper, the angular dependence is separated out giving rise to a system of 16 integral equations in one variable, which for $j=0$ is equivalent to a system of four equations and for $j>0$, eight. *A. J. Coleman.*

Chartres, B. A.; and Messel, H. Angular distribution in electron-photon showers without the Landau approximation. Phys. Rev. (2) **104** (1956), 517–525.

Hitherto, the track-length angular distribution of electrons in an electron photon cascade has been calculated for a number of models of the process but only in the Landau approximation. Using Hankel transforms the authors succeed in obtaining an exact expression for the distribution for the Tamm-Belenky model. They show that the Landau approximation is greatly in error even for small angles, contrary to expectations based on certain physical arguments. In the example graphed in the paper, the error is as much as thirty percent. Their exact solution, expressed in terms of the modified Bessel function of the second kind, is expanded in series. The first term is of the same form as results from the Landau approximation but greatly improved in accuracy due to a different choice of a parameter of the theory. An incidental result is the author's conclusion that the paper "shows the danger of using purely physical arguments to justify mathematical approximations". *A. J. Coleman.*

Jones, H. The specific heat of metals and alloys at low temperatures. Proc. Roy. Soc. London. Ser. A. **240** (1957), 321–332.

The elementary quantum-mechanical theory of conduction electrons in metals leads to the formula $C_v=\gamma T$ for the specific heat, C_v, when the Kelvin temperature, T, is small. The constant, γ, can be calculated in absolute terms, and gives a measure of the density of electronic energy states at the Fermi energy. In certain materials this formula appears to be defective in that the theoretical value of γ is considerably too small.

The author advances the theory that when the Fermi energy lies close to Brillouin zone boundaries the coupling between the conduction electrons and the lattice vibrations becomes important, and is sensitive to lattice deformations. This interaction is interpreted to lead to a slight temperature dependence of the Fermi energy of the electrons, which effectively increases the value of γ.

The theory is developed from this point in a semi-

phenomenological manner. Starting with the assumption that the energies of the lattice vibrations (phonons) depend slightly on the electronic energy distribution, the free energy of the lattice vibrations is calculated and used to find the temperature dependence of the elastic constants. This leads to an evaluation of the lattice energy which gives a specific heat of the form $C'=\gamma' T$, where the constant γ' depends sensitively on the proximity of the Fermi energy surface to the zone boundaries, and theoretically can become much larger than γ.

{This theory is of considerable interest for the clear manner in which the author has approached the problem, although no claim is made as to the ultimate adequacy of the mathematical formulation.} *E. L. Hill.*

Jánossy, L.; and Kiss, D. On the statistics of the determination of the mean life of μ-mesons. Acta Phys. Acad. Sci. Hungar. 7 (1957), 107–110. (Russian summary)

Lučina, A. A. On longitudinal oscillations of a plasma. I. Ž. Eksper. Teoret. Fiz. 28 (1955), 17–27. (Russian)

Myakišev, G. Ya.. and Lučina, A. A. On longitudinal oscillations of a plasma. II. Ž. Eksper. Teoret. Fiz. 28 (1955), 28–37. (Russian)

See also: Flodmark, p. 69; Kalicin, p. 76; Cohen, De Boer, and Salsburg, p. 78; Salsburg, Cohen, Rethmeier, and De Boer, p. 78; Iwamoto and Yamada, p. 79; Pyle, p. 84; Borgardt, p. 96; Born, p. 103; Kalitzin, p. 103.

Relativity

Born, Max. Physik und Relativität. Naturwiss. Rundschau 9 (1956), 417–424.

An account, given as a lecture in Bern in 1955, of Einstein's life and work, particularly up to the year 1930.

Libois, P. Actualité de la conception riemanienne de l'espace. Bull. Soc. Math. Belg. 8 (1956), 31–42.

An expository lecture with six sections: les espaces physiques, espace et temps, corps solide et objet, onde et corpuscule, connexion, les espaces.

Tauber, Gerald E. The gravitational fields of electric and magnetic dipoles. Canad. J. Phys. 35 (1957), 477–482.

Fields with axial symmetry are considered within the framework of classical relativistic electromagnetic theory. In canonical form the line element for such a field has the form

$$ds^2=e^{\rho}(dx^4)^2-e^{\lambda}((dx^1)^2+(dx^2)^2)-e^{-\rho}(x^2)^2(dx^3)^2.$$

By extending the work of Curzon [Proc. London -Math. Soc. (2) 23 (1924), 477–480] the author obtains solutions of the field equations that can be interpreted in terms of electric and magnetic dipoles. *M. Wyman.*

Penfield, Robert; and Zatzkis, Henry. The relativistic linear harmonic oscillator. J. Franklin Inst. 262 (1956), 121–125.

It is shown how, when the mass of a harmonic oscillator varies with velocity as in the special theory of relativity, the equations of motion may be integrated in terms of elliptic functions. The manner in which the period depends not only on the amplitude but the spring constant and the rest mass is also discussed. A recent publication dealing with the same problem is, as acknowledged in the paper, Møller, Danske Vid. Selsk. Mat.-Fys. Medd. 30 (1955), no. 10 [MR 17, 675]. *D. C. Lewis* (Baltimore, Md.).

Schmutzer, Ernst. Zur relativistischen Elektrodynamik in beliebigen Medien. Ann. Physik (6) 18 (1956), 171–180.

The constitutive equations of electromagnetic theory for a general anisotropic medium are put into a covariant form with respect to Lorentz transformations. The treatment differs from that of G. Marx [Acta Phys. Acad. Sci. Hungar. 3 (1953, 75–94; MR 15, 185]. A variational principle for the Maxwell equations is set up for an arbitrary medium. Arguments are presented in favor of the Minkowski form of the energy-momentum tensor in a material medium, and against that of Abraham.
N. Rosen (Haifa).

Kantor, Wallace; and Szekeres, George. Cosmic time and the field equations of general relativity. Phys. Rev. (2) 104 (1956), 831–834.

Within the framework of a previous paper [G. Szekeres, Phys. Rev. (2) 97 (1955), 212–223; MR 16, 869] the authors explore the consequence of taking a different Lagrangian whose choice is justified by a gauge postulate. A static centrosymmetrical solution of the resulting field equations is obtained. An error in the expression for the gravitational potential given in the previous paper is corrected.
A. J. Coleman (Toronto, Ont.).

Marder, L. On uniform acceleration in special and general relativity. Proc. Cambridge Philos. Soc. 53 (1957), 194–198.

Two possible ways of generalizing to relativity theory the idea of uniform acceleration are suggested. Their consequences are explored for a body propelled by emitting mass particles or photons and for a body moving through a dust cloud. *A. J. Coleman* (Toronto, Ont.).

Freistadt, Hans. The significance of relativity. Invariance and covariance. Rev. Mexicana Fis. 5 (1956), 43–51. (Spanish. English summary)

Tensor covariance and the invariance of physical laws, as axioms of relativity theory, are distinguished and discussed. *N. Rosen* (Haifa).

Kalitzin, Nikola St. Über die sechsdimensionale Theorie des Mesonfeldes. Jbuch. Staatsuniv. Stadt Stalin Fak. Bauwesen 1 (1953), 127–142. (Bulgarian. German summary)

This is a continuation of an earlier work (in Bulgarian) in which wave equations are set up in a six-dimensional space with two time-like coordinates. The energy-stress tensor is calculated, and it is shown that the energy density of the meson field is positive. The interaction potential between two nucleons is found to be free from the usual $1/r^3$ singularity. *N. Rosen* (Haifa).

Kalitzin, Nikola St. Über eine Verallgemeinerung der allgemeinen Relativitätstheorie. Jbuch. Staatsuniv. Stadt Stalin Fak. Bauwesen 1 (1953), 143–150. (Bulgarian. German summary)

A Riemannian space is introduced, consisting of many sheets. Particles of a given ratio of electric charge to mass move on a given sheet. The formalism of the general relativity theory gives in first approximation the Lorentz force on the particle. *N. Rosen* (Haifa).

Davis, W. R. Über "starke" und "quasi-starke" Erhaltungssätze allgemein kovarianter Feldtheorien und ihre allgemeinen differenziellen Identitäten. Z. Physik **148** (1957), 1–14.

The author considers covariant field theories, where the Lagrangian is a function of two field quantities with different transformation properties. The general identities are derived (the generalized Bianchi-identities). This general theory is later applied to Einstein's unitary theory [The meaning of relativity, 4th ed., Princeton, 1953; MR **14**, 805; **15**, 357]. *L. Infeld* (Warsaw).

Ghosh, N. N. On a solution of field equations in Einstein's unified field theory. II. Progr. Theoret. Phys. **17** (1957), 131–138.

[For part I see same journal **16** (1956), 421–428; MR **18**, 704.] The author extends the known solutions of the strong form of the field equations in Einstein's Unified Field Theory by finding a non-static solution that is not spherically symmetric. *M. Wyman* (Edmonton, Alta.).

Krogdahl, Wasley S. Kinematic relativity and the operational formulation of a Lorentz invariant dynamics. Z. Astrophys. **42** (1957), 48–65.

There is some similarity between the ideas of this paper and those of Milne's "Kinematic relativity" [Oxford, 1948; MR **10**, 578]. The authors try to remove the logical incompleteness of the special relativity theory which they see in the fact that in it the analogue of the third law of motion is lacking. They also formulate a Lorentz-invariant theory of gravitation. It gives the proper perihelion advance when an appropriate Lorentz-invariant potential function is chosen. *L. Infeld* (Warsaw).

See also: Kuznecov, p. 95; Shirokov and Sannikov, p. 96; Arcidiacono, p. 108.

Astronomy

Morrison, Philip. On the origins of cosmic rays. Rev. Mod. Phys. **29** (1957), 235–243.

This excellent review article is a written version of a lecture delivered at the Seattle Conference on Theoretical Physics in 1956. It gives a lucid and entirely qualitative account of the present status of the theory of the origin of cosmic rays. The processes of stirring, storage and loss, acceleration, injection, and cutoff are treated in this order. The conclusion is that while some or even all the details of the picture might turn out to be wrong, we can be pretty certain that we have hit upon the basic ideas which will be able to give a rigorous and quantitative explanation of the origin of cosmic rays.

M. J. Moravcsik (Upton, N.Y.).

Pierucci, M. Sull'età dell'Universo. Nuovo Cimento (10) **5** (1957), 572–578.

In an earlier paper [Accad. Sci. Lett. Arti Modena. Atti Mem. **9** (1951), 274–284] the author discussed the concept of absolute time and gave a theoretical estimate for the age of the Universe. The previous hypotheses are now modified to incorporate certain astronomical observations, and a figure of 11×10^9 years is deduced for the age of the Universe. *A. G. Walker* (Liverpool).

See also: Lenoble, p. 93.

Geophysics

Lliboutry, L. La mécanique des glaciers en particulier au voisinage de leur front. Ann. Géophys. **12** (1956), 245–276.

Chopra, S. D. The range of existence of Stoneley waves in an internal stratum. I. Symmetric vibrations. Monthly Not. Roy. Astr. Soc. Geophys. Suppl. **7** (1957), 256–270.

A layer of constant thickness is bounded on both sides by identical half-spaces with which it is in welded contact. The two media are perfectly elastic, homogeneous, and isotropic. The author obtains a frequency equation of the fourth order for surface-type (Stoneley) waves, assuming the mid-plane of the layer to be a plane of symmetry of the motion. The wave velocities studied here are less than those of distortional waves in either medium, and Poisson's ratio is taken to be $\frac{1}{4}$ in both media. The equation has then at most one relevant root. Computations are presented in the form of a table and of curves. It is found that symmetrical waves of the type discussed here can exist in only very limited circumstances. *F. Ursell.*

Kuzivanov, V. A. On analytic continuation of gravitational potential into an interior region. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 1419–1426. (Russian)

Discussing the determination of the depth to the singularities of gravity potential by analytical continuation downwards of mapped values of its vertical gradient, the author shows by many examples its practical impossibility caused by the fact that this method is extremely sensitive to small errors in the observed values. It is also pointed out that a detailed knowledge of tectonic structure and of density distribution in the masses of the earth's crust lying between the surface of the ground and the geoid is necessary to perform a correct reduction of observed gravity to the geoid's surface.

E. Kogbetliantz (New York, N.Y.).

Kuzivanov, V. A. On the question of reduction of an anomalous gravitational force. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 1161–1173. (Russian)

Study of a new method for correct reduction of observed gravity values to a datum-plane taking into account the exact configuration of the earth's surface. This method is based on the equations which characterize the quasigeoid of Molodewski and which were deduced by Molodewski. *E. Kogbetliantz* (New York, N.Y.).

Strahov, V. N. Some questions of method in the interpretation of magnetic anomalies. I. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 1389–1399. (Russian)

This paper deals with the interpretation of magnetic profiles $H=H(x)$, $Z=Z(x)$ mapped along the OX-axis and caused by an infinite horizontal cylindrical structure of normal section S in the plane OXY (OY is the vertical) and of magnetization $Ie^{i\gamma}=I_x+iI_y$ (two dimensional case). To find γ, IS as well as the depth to the center of gravity of S and its shape the two profiles are combined into a vector $H(x)-iZ(x)$ in the complex plane OHZ so that a closed curve Γ_x (called "indicatrix") is obtained in the plane OHZ. Another curve Λ_x ("logarithmic indicatrix") is obtained in the plane $OT\varphi$ with the aid of the vector $\omega=T+i\varphi=\log[H(x)-iZ(x)]$.

The interpretation is based on a collection of theoretical curves Γ_x and Λ_x precomputed for various ca-

nonical shapes of S and various depths. The shape and orientation of Γ_x or Λ_x are compared with curves of these collections.

{The mathematical part based on the theory of conformal mapping is correct, but the interpretations based on comparisons of experimental curves with theoretical precomputed cases are not reliable in general. The author recommends the use of Λ_x as preferable to the use of Γ_x, but it seems that even small errors in the choice of base lines H_0, Z_0 which represent normal values of earth magnetic field should cause wrong interpretation.}

E. Kogbetliantz (New York, N.Y.).

Ansermet, A. L'extension au cas de mesures linéaires d'un théorème de Schreiber. Schweiz. Z. Vermessg. Kulturtech. Photogr. **55** (1957), 33–38.

Dans les réseaux determinés en fonction de mesures angulaires un problème fut posé, il y a longtemps dejà, tendant à répartir les poids de façon favorable lorsque la somme de ces poids est une constante. Un problème analogue peut se poser dans le cas de mesures linéaires. Le but de ces lignes est de formuler quelques considérations à ce sujet. *Résumé de l'auteur.*

Tárczy-Hornoch, A. Zur Ausgleichung der kontinentalen Triangulierungsnetze. Acta Tech. Acad. Sci. Hungar. **16** (1957), 429–435. (English, French, and Russian summaries)

In a large 1st order triangulation scheme, the precision of adjustment can be increased by introducing side lengths as well as angles into the computation, since the number of excess observations will be larger. If these side lengths can be directly measured, then the method of the variation of coordinates (indirect observations) will preserve the added precision, and at the same time reduce the number of normal equations in the computation.

B. Chovitz (Washington, D.C.).

Mineo, Corradino. Ancora sulla geodesia intrinseca. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 552–555.

A continuation of the article listed in MR **17**, 1128.

See also: Rosenblatt, p. 87; Lowndes, p. 92; Halfin, p. 92; Lenoble, p. 93.

OTHER APPLICATIONS

Economics, Management Science

★ **Gale, David. The closed linear model of production.** Linear inequalities and related systems, pp. 285–303. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

Von Neumann's model of a linear economic system expanding at a constant rate [Erg. Math. Kolloq. **8** (1937), 73–83] has recently been further studied and somewhat generalized by Georgescu-Roegen [Activity Analysis of Production and Allocation, Wiley, New York, 1951, pp. 98–115; MR **13**, 262], the reviewer and Samuelson [Econometrica **21** (1953), 412–424; MR **15**, 49], Kemeny, Morgenstern and Thompson [ibid. **24** (1956), 115–135; MR **18**, 266] and others. Von Neumann's original paper included an extension of Brouwer's fixed-point theorem, and one of the achievements of Gale's paper is to eliminate topological notions from the discussion altogether. The main theorem is made to rest essentially on the property of a convex cone that if it is not the whole space it is contained in a halfspace. By contrast it has not proved possible to eliminate topology from Wald's theorem [see Kuhn, Linear inequalities and related systems, Princeton, 1956, pp. 265–273; MR **18**, 269], presumably because of the essential complication introduced by the demand functions f.

The generalization of von Neumann's model contained herein consists of letting the set of achievable $2n$-dimensional input-output vectors (x, y) be a closed convex cone. In von Neumann's paper the cone was restricted to be polyhedral. In addition, von Neumann's restriction that every commodity should figure either as input or output in each process is replaced by a weaker condition of "regularity", that for every optimal process (i.e. expanding at the highest achievable rate) the output vector be strictly positive. The von Neumann case is discussed in detail, and then specialized further to Leontief models, namely those for which there are as many basic processes as commodities, and each basic process has one output. The results here are already known, but are very simply proved. *R. Solow* (Cambridge, Mass.).

Seng, You Poh. Some theory of index numbers. II. Maintenance of continuity by the assumption of proportionality of the new index number to the old index. J. Roy. Statist. Soc. Ser. A. **119** (1956), 425–455.

This is the second of two articles, the first of which has been reviewed previously [same J. Ser. A. **119** (1956), 312–332; MR **18**, 629]. The purpose of the study is to investigate the questions arising when a somehow changed index number sequence has to be compared with the corresponding numbers obtained if these changes had not been made. The "errors" are discussed by means of a lot of mathematical formulas and by numerical examples showing the range of discrepancies resulting in "errors" of a certain order of size. *L. Törnqvist* (Helsinki).

Enthoven, Alain C.; and Arrow, Kenneth J. A theorem on expectations and the stability of equilibrium. Econometrica **24** (1956), 288–293.

Let P_i and P_i' be, respectively, the current and the expected price of the ith good. Let $\dot{P}_i$=rate of change of P_i, and X_i=excess of demand over supply. In equilibrium, for every i, $X_i=0$, $P_i'=P_i=P_i^0$, say. Assume, for all i,

$$P_i'=P'+\eta_iP_i;\ \dot{P}_i=K_iX_i,\ K_i>0.$$

In the neighborhood of the equilibrium, approximately

$$X_i=\sum_j a_{ij}(P_j-P_j^0)+b_i(P_i-P_i^0).$$

Hence

$$\dot{P}_id_i/K_i=\sum_j a_{ij}(P_j-P_j^0)+b_i(P_i-P_i^0),$$

where $d_i=(1-K_ib_i\eta_i)^{-1}$. If all $\eta_i=0$ (the case of "static expectations") and all $a_{ij}>0$ $(i\neq j)$ (i.e., "all goods are gross substitutes"), then the system possessses a stable equilibrium if and only if the matrix

$$A=\begin{bmatrix} K_1(a_{11}+b_1) & \cdots K_1a_{1n} \\ \cdot & \cdots \\ K_na_{n1} & \cdots K_n(a_{nn}+b_n)\end{bmatrix}$$

is negative definite. This result due to L. A. Metzler [Econometrica **13** (1945), 277–292; MR **7**, 465] is extended

to the case when some $\eta_i \neq 0$, provided all $d_i > 0$, i.e., provided that, for each good, the quantity $1/K_i$ (the "insensitivity of price to excess demand") is greater than $b_i\eta_i$ ("the destabilizing force of the extrapolative expectation"). To obtain this new result, the following theorem had to be proved: If A has all negative diagonal elements, and no negative off-diagonal elements, if D is a diagonal matrix, and if the real parts of the characteristic roots of both A and DA are negative, then the diagonal elements of D are positive. *J. Marschak.*

Klein, L. R. The scope and limitations of econometrics. Appl. Statist. **6** (1957), 1–17.

Author first considers in general the way in which econometric techniques can be developed to estimate the interrelationships among economic variables, and then deals in some detail with applications of econometrics to particular problems. *Author's summary.*

Banerjee, K. S. A note in the treatment of composite items in the construction of cost of living index numbers. Calcutta Statist. Assoc. Bull. **7** (1956), 35–40.

Biser, Erwin; and Meyerson, Martin. The application of design of experiments and modeling to complex weapons systems. Operations Res. **5** (1957), 210–221.

See also: Basmann, p. 74.

Programming, Resource Allocation, Games

Saaty, Thomas L. Résumé of useful formulas in queuing theory. Operations Res. **5** (1957), 161–200.

This assembly of results in queuing theory is intended for the use of workers in operations research. As such it concentrates on the simpler measures of system effectiveness, the equilibrium state probabilities and the means and variances of queue length and waiting time. These results are given, where possible, for Poisson and regular arrivals, for single and multiple channels with exponential, Erlangian and constant service time, and for order of arrival, priority, and random choice of items from the queue, which in one case is allowed defections. There are historical remarks, some indication of the structure of the theory, and a bibliography of 68 items (many of which go beyond the summary). Derivations are purposely omitted and there is little attention to tables and curves for numerical evaluation. A number of typographical and other slips appear; of these it may be noted that equations (22) and (23) should read $L=\rho(\rho+2k-\rho k)/2k(1-\rho)$ and $W=\rho(k+1)/2(1-\rho)k\mu$ with $\rho=\lambda/\mu$. {It is somewhat surprising that the "busy period," the time of full occupancy of channels, which is also a measure of effectiveness, is ignored, that the many channel state probabilities for constant service time are omitted, and that the bibliography contains only one item for Conny Palm and none for Fortet, Kosten and Vaulot.} *J. Riordan.*

Markowitz, Harry M.; and Manne, Alan S. On the solution of discrete programming problems. Econometrica **25** (1957), 84–110.

The authors apply methods of the type used by Dantzig, Fulkerson and Johnson in solving traveling salesmen problems [J. Operations Res. Soc. **2** (1954), 393–410; MR **17**, 58] to some linear programming problems in which solutions are required to be integral. A detailed description is given of the specific computations. It is pointed out that the method makes essential use of human ingenuity (as described here it would, for instance, be out of the question to attempt coding for automatic computation.) The authors also show how certain problems involving increasing returns to scale can be approximated by discrete linear problems. *D. Gale.*

Kay, Emil; and Duckworth, Eric. Linear programming in practice. Appl. Statist. **6** (1957), 26–39.

The simplex method is used to determine the least-cost proportions of new and recovered metal in making up alloys to specification. *W. F. Freiberger.*

Cansado, Enrique. Linear programming, a mathematical instrument in the service of contractors. Trabajos Estadist. **7** (1956), 305–335 (3 plates). (Spanish)

An expository paper on linear programming.

Rohde, F. Virginia. Bibliography on linear programming. Operations Res. **5** (1957), 45–62.

Luce, R. Duncan; and Rogow, Arnold A. A game theoretic analysis of congressional power distributions for a stable two-party system. Behavioral Sci. **1** (1956), 83–95.

The model is a game of $n+1$ players, n players in Congress plus the President. A winning set is a congressional majority plus the President, or a two-thirds majority. For several values of τ and ψ the authors examine imputations x such that (x, τ) is ψ-stable in the sense of Luce [Mathematical models of human behavior, 1955, pp. 32–44]. The basic restriction is that τ is a partition into two parties, C_1, C_2, and that ψ is as follows: each party C_i consists of a set of diehards C_i'' who will never leave the coalition and a remainder C_i' who may leave in a block; however, the coalition $C_1' \cup C_2'$ is not permitted. For the 36 cases, the conclusions on x agree roughly with the findings of political scientists, and appear to indicate directions for further work. *J. Isbell.*

Zięba, A. On the pursuit. Prace Mat. **2** (1956), 117–130. (Polish. Russian and English summaries)

Strategies and minimax solutions are investigated in some cases of a continuous pursuit game considered by H. Steinhaus [Definitions necessary for the theory of games and pursuit, Lwów, 1928]. No reference is made to the work of J. von Neumann and O. Morgenstern [Theory of games and economic behavior, 2nd ed., Princeton, 1947; MR **9**, 50], and the terminology differs from that established in their treatise. *S. K. Zaremba.*

Zięba, A. An elementary proof of von Neumann's minimax theorem. Colloq. Math. **4** (1957), 224–226.

See also: Ehrenfeucht, p. 4; Bellman, p. 5.

Biology and Sociology

Burke, C. J.; and Estes, W. K. A component model for stimulus variables in discrimination learning. Psychometrika **22** (1957), 133–145.

In this learning model "stimulus situations" are assembled from sets of available stimulus elements in accord with probability distributions over these elements. The model is to learn to respond differently to the situa-

tions derived from different distributions. Each stimulus element is "connected" at any time to one of two possible responses; the probability of occurrence of each response is proportional to the number of occurring stimulus elements connected to it at the time. Reinforcement operates on the probabilities of these connections.

Relations are found between the amount of overlapping of two stimulus distributions and the rates and limits of discrimination learning. Some results are also obtained in the case that the reinforcement has an effect on the subsequent stimulus. *M. L. Minsky.*

See also: Bennett, p. 16.

Information and Communication Theory

Prihar, Z. Topological properties of telecommunication networks. Proc. I.R.E. **44** (1956), 927–933.

This paper describes a method of matrix analysis developed by R. D. Luce and A. D. Perry [Psychometrika **14** (1949), 95–116; MR **12**, 39] for the study of sociometric group structures and cliques. This method together with certain extensions, is applied to radio and cable communication. *R. J. Duffin* (Pittsburgh, Pa.).

See also: Peterson, p. 69; Rozenblat-Rot, p. 71.

Control Systems

Kadymov, Ya. B. On methods of study of stability of systems of automatic regulation with distributed parameters. Akad. Nauk Azerbaĭdžan. SSR. Dokl. **12** (1956), 543–545. (Russian. Azerbaijani summary)

This note is concerned essentially with the stability of linear systems with retardation. The author lists several types of characteristic equations for such systems and, by way of illustration, discusses the equation

$$\varphi_1(z)e^{z\tau}+\varphi_2(z)e^{-z\tau}+\varphi_3(z)=0,$$

where $\varphi_1(z)$, $\varphi_2(z)$ and $\varphi_3(z)$ are real functions of the complex variable z, and τ is a constant parameter. However, his analysis is incomplete and the arguments are not convincing to this reviewer. *L. A. Zadeh.*

Ostrovskiĭ, G. M. Application of nonlinear constructions in systems of automatic regulation of the third order. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **11** (1956), no. 1, 51–56. (Russian)

This paper is concerned with the problem of synthesis of a control system in which the error variable, x, satisfies a third-order equation of the form

$$\dddot{x}+f_1(x)\ddot{x}+f_2(x)\dot{x}+x=0.$$

The functions $f_1(x)$ and $f_2(x)$ are assumed to be subject to the conditions

$$a_1\leqq f_1(x)\leqq b_1,\quad a_2\leqq f_2(x)\leqq b_2,$$

where a_1, b_1, a_2 and b_2 are given constants. The problem is to determine $f_1(x)$ and $f_2(x)$ which (i) result in no overshoot in the response to a unit-step disturbance, and (ii) minimize the length of time during which the error exceeds a prescribed constant.

Let $-\gamma_1$, $-\gamma_2$, $-\gamma_3$ $(\gamma_1<\gamma_2<\gamma_3)$ be the roots of the characteristic equation

$$\rho^3+b_1\rho+b_2\rho+\rho=0,$$

and let

$$\Phi(x,\dot{x},\ddot{x})=[\gamma_2\gamma_3x+(\gamma_2+\gamma_3)\dot{x}+\ddot{x}]\dot{x}.$$

By a fairly involved and not quite self-contained analysis, the author obtains the following expressions for $f_1(x)$ and $f_2(x)$:

$$f_1(x)=\begin{cases}a_1 \text{ for } \Phi(x,\dot{x},\ddot{x})<0,\\ b_1 \text{ for } \Phi(x,\dot{x},\ddot{x})\geqq0,\end{cases}$$

$$f_2(x)=\begin{cases}a_2 \text{ for } \Phi(x,\dot{x},\ddot{x})<0,\\ b_2 \text{ for } \Phi(x,\dot{x},\ddot{x})\geqq0.\end{cases}$$

L. A. Zadeh (New York, N.Y.).

Petrov, V. V.; and Ulanov, G. M. Common properties of gliding, vibrational and optimal regimes of a class of servomechanisms. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 394–397. (Russian)

The three modes of operation of a relay servomechanism considered in this paper have the following significance. By the gliding or pulsating regime is meant a mode of operation where the output of the relay consists of a series of pulses having the same sign; the vibrational regime represents the case where the relay is part of an inner feedback loop containing an integrator; and the optimal regime is defined as one where the equilibrium is approached as rapidly as possible and without overshoot. The authors point to certain similarities in the dynamic characteristics of these regimes and discuss, in qualitative terms, the conditions under which a gliding regime approaches an optimal regime. *L. A. Zadeh.*

See also: Frid, p. 34; Tusov, p. 34; Worthy, p. 68.

HISTORY, BIOGRAPHY

★**Becker, Oskar. Das mathematische Denken der Antike.** Vandenhoeck and Ruprecht, Göttingen, 1957. 128 pp.

Carefully selected passages give not only the mathematical specialist but also the layman a view of the chief results and of the methods of proof of ancient mathematics. Part One contains a short historical introduction to Egyptian, Babylonian and especially Greek mathematics. Part Two gives passages, with commentary, from such sources as the Rhind papyrus, the Cuneiform texts, the pre-Socratics, Euclid, Archimedes, Diophantus. The characteristically Greek application of areas and the rigorous foundation of proportionality by Eudoxus are well treated. As the author points out in appendix, the extensive systematic theories of the golden age as found in the works of Euclid, Archimedes and Apollonius could not be adequately represented. Nevertheless the book will give its readers precise and representative ideas about Greek mathematics.

Noguera Barreneche, Rodrigo. Two identities with historical antecedents. Stvdia. Rev. Univ. Atlantico **1** (1956), no. 10, 63–72. (Spanish)

Two identities which appear to underlie many of the equations in the Rhind papyrus.

★**Hofmann, Joseph Ehrenfried. The history of mathematics.** Philosophical Library, New York, 1957. xi+132 pp. $4.75.

A translation by Frank Gaynor and Henrietta O.

Midonick from the German of the book reviewed in MR 15, 275.

Hofmann, Jos. E. Zur Entwicklungsgeschichte der Eulerschen Summenformel. Math. Z. 67 (1957), 139–146.

A discussion, based on some still unpublished correspondance of Euler, of the way in which he may have come upon the formula nowadays called the Euler-Maclaurin summation formula, and also of Euler's general attempts to replace the sum of a series by a definite integral and vice versa.

Cremona, Luigi. Le figure reciproche. Civilità delle Macchine 4 (1956), no. 5, 55–62.

A reprinting, with a biographical introduction, of the work originally published, in a de luxe volume together with an article by Casorati, in 1872.

Blaschke, Wilhelm; und Schoppe, Günther. Regiomontanus: commensurator. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. 1956, 445–529.

A complete edition in German translation with commentary of the work subtitled "Geometric problems of every kind", the manuscript of which came to attention only in 1938. The present edition should be of great importance for the history of mathematics.

de Mira Fernandes, A. An ephemeris. On the centenary of the geometry of Riemann. Univ. Lisboa. Revista Fac. Ci. A. (2) 5 (1956), 329–342. (Portuguese)

Historical survey of the geometry of Riemann and of its modern generalizations. *L. A. Santaló.*

Natucci, Alpinolo. Che cos'è la "trauagliata inventione" di Nicolò Tartaglia? Period. Mat. (4) 34 (1956), 294–297.

A brief account of the method proposed by Tartaglia for raising sunken ships and of its connection with the works of Archimedes.

Lebesgue, Henri. Sur une construction du polygone régulier de 17 cotés, due a André-Marie Ampère, d'après des documents conservés dans les Archives de l'Académie des Sciences. Enseignement Math. (2) 3 (1957), 31–34.

Steinhaus, Hugo. The collaboration of various sciences, as illustrated by mathematics, and its role in Wroclaw scientific circles. Rev. Polish Acad. Sci. 1 (1956), no. 4, 1–20.

Milankovitch, M. Aristarchos und Apollonios. Das heliozentrische und das geozentrische Weltsystem des klassischen Altertums. Acad. Serbe Sci. Publ. Inst. Math. 9 (1956), 79–92.

A detailed argument is given that the original motivation of the epicyclic theory of Apollonius of Perga lay entirely in his desire to represent, as it would be seen from the earth, the motion of the planets according to the heliocentric theory of Aristarchus of Samos.

S. H. Gould (Providence, R.I.).

★ **Piccard, Sophie. Lobatchevsky, grand mathématicien russe: sa vie, son oeuvre.** Université de Paris, Paris, 1957. 39 pp.

A biography in popular style with one photograph and a bibliography of 25 works by or about Lobatchevsky.

Lefschetz, Solomon. Witold Hurewicz, in memoriam. Bull. Amer. Math. Soc. 63 (1957), 77–82.

A scientific biography with a list of 49 publications.

Denjoy, Arnaud; Felix, Lucienne; et Montel, Paul. Henri Lebesgue, le savant, le professeur, l'homme. Enseignement Math. (2) 3 (1957), 1–18.

Lebesgue, Henri. Notice sur René-Louis Baire, correspondant pour la section de géométrie. Enseignement Math. (2) 3 (1957), 28–30.

Martín Jadraque, V. Delambre. Gac. Mat., Madrid (1) 8 (1956), 191–193. (Spanish)

A general biographical note.

Chandrasekharan, K. Obituary: T. Vijayaraghavan. Math. Student 24 (1956) 251–267 (1957).

A general and scientific notice with a list of 31 mathematical articles and 13 shorter notes.

Courant, Richard. Franz Rellich zum Gedächtnis. Math. Ann. 133 (1957), 185–190.

A short scientific biography, with a photograph and a list of 26 published works, together with a list of lectures delivered in Göttingen and elaborated into lecture notes.

Edge, W. L. Obituary: H. F. Baker, F. R. S. Edinburgh Math. Notes no. 41 (1957), 10–28.

A general and scientific notice, with a complete list of Baker's books and a partial list of his articles.

Pellegrino, Franco. In memoria di Luigi Fantappiè. Rend. Mat. e Appl. (5) 15 (1956), 505–519 (1957).

A scientific biography with a list of 106 publications.

Erdélyi, A. Sir Edmund Whittaker, 1873-1956. Math. Tables Aids Comput. 11 (1957), 53–54.

A short notice of Whittaker's importance in modern computation.

Anonymous. Obituary: Herbert Emil Ludwig Bilharz. Arch. Math. 8 (1957), i (1 plate).

A notice of the importance of Bilharz as editor and as mathematician, with one photograph.

Tompkins, C. B. John von Neumann, 1903-1957. Math. Tables Aids Comput. 11 (1957), 127–128.

Orts, José Maria. Fantappiè and analysis. Rev. Mat. Hisp.-Amer. (4) 17 (1957), 3–9. (Spanish)

Arcidiacono, Giuseppe. Fantappiè e la relatività. Rev. Mat. Hisp.-Amer. (4) 17 (1957), 14–17.

★ **Carathéodory, Constantin. Gesammelte mathematische Schriften. Bd. 5.** Herausgegeben im Auftrag und mit Unterstützung der Bayerischen Akademie der Wissenschaften. C. H. Beck'sche Verlagsbuchhandlung, München, 1957. xv+447 pp. (1 plate). DM 43.00.

This volume completes the beautiful edition of C. Carathéodory's collected mathematical papers [see the reviews of the volumes I–IV in MR 16, 434, 985; 17, 446; 18, 453]. Volume V begins with four papers on geometry. Three of them deal with differential geometry. The fourth one is a short note "Beweis eines Satzes über Parallelkörper" [2 pp.] published here for the first time;

it is an except of a letter to E. Schmidt concerning one of his articles [Math. Nachr. **1** (1948), 81–157, in particular p. 96; MR **10**, 471] in which the boundary of the „Parallelkörper" is proved to have volume zero, while Carathéodory here shows that its surface is finite. Volume V continues with two papers on partial differential equations, 16 articles of a historic-biographic nature (among them two previously unpublished short articles and the remarkable introduction [68 pp.] to Euler's calculus of variations [Opera omnia, ser. 1, v. 24, 25, Soc. Sci. Nat. Helv., Bern, 1952; MR **15**, 89]). Then there are 12 essays of various kinds (five of which were translated from Greek into German by C. Carathéodory's son Stephanos) and three posthumous notes: "Das einfache Pendel für große Anfangsgeschwindigkeiten" [4 pp.]; "Auszug aus dem Vorwort zu Reelle Funktionen Band II: Die Theorie des Integrals" [instead of this volume II, Carathéodory's book "Maß und Integral und ihre Algebraisierung" was published by Birkhäuser, Basel-Stuttgart, 1956; MR **18**, 117]; „Länge und Oberfläche" [a short outline of Carathéodory's last lecture on December 16, 1949; a facsimile of this handwritten page is added]. There follow 30 reviews and very interesting and colorful autobiographical notices [20 pp.] which, unfortunately, are not complete, but cover only the first 35 years of Carathéodory's life, up to 1908, the end of his activity as Privatdozent at Göttingen University. This volume concludes with a thoughtful necrology [9 pp.] written by his special friend Erhard Schmidt, and a list of all publications of Carathéodory in historical order. A photograph of Carathéodory as a young engineer is also included.

The five volumes of collected mathematical papers show, in a very impressive manner, the outstanding contributions of Carathéodory to so many different fields of mathematics and the abundance of his original ideas. In the order of these volumes, his papers deal with calculus of variations, thermodynamics, geometrical optics, mechanics, analytic functions of one and of several complex variables including conformal mapping, real functions, in particular measure theory and algebraisation of the integral, and the above mentioned contents of volume V. Quite a few of these subjects were also systematically treated in the various remarkable, classical books of Carathéodory: Vorlesungen über reelle Funktionen [Teubner, Leipzig-Berlin, 1918, 2nd ed. 1927], which have greatly influenced a whole generation of mathematicians, and, closely related to them, Reelle Funktionen, vol. I Teubner, Leipzig, 1939; MR **1**, 205]; Conformal representation [Cambridge, 1932, 2nd ed. 1950; MR **13**, 734]; Variationsrechnung und partielle Differentialgleichungen erster Ordnung [Teubner, Leipzig-Berlin, 1935]; Geometrische Optik [Springer, Berlin, 1937]; Funktionentheorie [2 vols., Birkhäuser, Basel, 1950; MR **12**, 248]; Maß und Integral und ihre Algebraisierung [vide supra]. The existence of one book, "Syllogos ophelimon biblion Ar. 14" of Carathéodory (quoted in the above mentioned list of publications) has probably been unknown even to most of his friends; it is a book on Egypt (written in Greek), published in Athens, 1901 [2nd ed. 1926, reprinted in New York, 1920]. *A. Rosenthal.*

★ **Ляпунов, А. М. [Lyapunov, A. M.] Собрание сочинений. Том I. [Collected works. Vol. I.]** Izdat. Akad. Nauk SSSR, Moscow, 1954. 447 pp. (4 plates). 22 rubles.

The first volume includes 7 articles on potential theory, 6 on probability, 1 on series, 4 on hydrostatics and hydrodynamics, and 3 on theoretical and celestial mechanics. There is a biographical essay with 3 photographs.

★ **Bianchi, Luigi. Opere. Vol. V. Trasformazioni delle superficie e delle curve.** Edizioni Cremonese, Roma, 1957. 538 pp. 5000 Lire.

The present volume contains 25 articles, with an introduction by Pietro Tortorici.

See also: Jasin'ski, p. 81; Wierzbicki, p. 81; Jasin'ski, p. 81.

MISCELLANEOUS

★ **Oeuvres de Henri Poincaré.** Publiées sous les auspices de l'Académie des Sciences par la Section de Géométrie. Tome XI. Publié avec la collaboration de Gérard Petiau. Gauthier-Villars, Paris, 1956. 305 pp.

Dans les tomes I à X des Oeuvres de Henri Poincaré a été inséré l'ensemble des articles, notes, mémoires, à caractère scientifique et classés par Ernest Lebon dans sa Bibliographie analytique des écrits de Henri Poincaré dans les sections Analyse mathématique, Mécanique analytique et Mécanique céleste, Physique mathématique. Figurent reproduits ci-dessus dans la première partie du tome XI quelques textes parmi les plus importants des publications de Henri Poincaré classés dans la Bibliographie d'E. Lebon dans les sections Philosophie scientifique (articles, discours, conférences), Histoire des Sciences (discours nécrologiques, articles et notices nécrologiques, discours, rapports, articles, préfaces, analyses), Publications diverses (notes, articles, conférences, discours, rapports, préfaces, analyses). Nous y avons ajouté en outre les correspondances entre Henri Poincaré et Mittag-Leffler, L. Fuchs et F. Klein publiées dans les tomes 38 et 39 des Acta mathematica. [From Notes, p. 134 of vol. XI.]

In addition to the Mémoires divers cited above, the section entitled Hommages à Henri Poincaré contains essays by P. Appell, M. Mittag-Leffler, J. Hadamard, W. Wien, H. A. Lorentz, H. von Zeipel, and Max Planck. The debt of astronomy to Poincaré is discussed at length (von Zeipel), and Hadamard's L'oeuvre mathématique de Poincaré is quite extensive.

The volume concludes with "Le livre de centenaire de la naissance de Henri Poincaré", which contains various lectures delivered during the centenary celebrations and over 50 pages of photographs, reproductions of letters, reports and various documents.

★ **Anonymous. Scientific and technical translating and other aspects of the language problem.** UNESCO, 19, Avenue Kléber, Paris, 1957. 282 pp. $4.20.

This report, as is stated in the introduction, deals with the general problem of enabling scientists to exploit knowledge published in languages other than their own. The greater part of its extensive factual information and expert opinion is not of interest directly to mathematicians, at least under present-day circumstances. But even the mathematical specialist will be helped by various remarks, for example, about mathematical literature in Russian and Chinese.

★ **Wayland, Harold. Differential equations applied in science and engineering.** D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto -New York-London, 1957. xiii+353 pp. $7.50.

This text, designed for upper classmen and beginning graduate students in engineering and applied science is distinctive for two reasons. In the first place, much attention is given to the formulation of the mathematical model, while secondly, the exposition is directed to the solution of linear partial differential equations in which the variables are separable. Thus there is a consistent orientation which unifies the different topics covered. The level of mathematical rigor is that of elementary calculus, but the necessary existence theorems are usually stated. The level of maturity regarding physical concepts is more sophisticated. The book is a welcome addition and should serve as a sound introduction to methods much used in applications. The typography is excellent and no errors were noted.

Chapter II is an intuitive brief treatment of vector analysis in rectangular and other coordinate systems. Chapter III gives a quick survey of ordinary differential equations, including Green's functions. Chapter IV is devoted to power series solutions at ordinary and regular points. Chapter V discusses functions defined by series solutions of differential equations. Chapter VI, on Fourier series and orthogonal functions, has a nice treatment of the vibrating string. Chapter VII, on boundary value problems, is the heart of the book. The discussion on problem formulation should help the student to get perspective on what is being done. Chapter VII is a brief introduction to integral transforms. *M. E. Shanks.*

★ **Myklestad, N. O. Fundamentals of vibration analysis.** McGraw-Hill Book Co., Inc., New York-Toronto-London, 1956. viii+260 pp. $6.50.

This book, though undoubtedly of value in the training of engineers, is of no significance to the mathematician interested in mechanics. The most advanced mathematical topics are, say, Lagrange's equations and the numerical solution of algebraic equations. One finds missing even some of the more sophisticated (though well-known) methods for the numerical approximation of frequencies, the Rayleigh method being the only one included. The book contains a large number fo examples, both of those solved in the text and of those left to the reader.

D. C. Lewis, Jr (Baltimore, Md.).

McShane, E. J. Maintaining communication. Amer. Math. Monthly 64 (1957), 309–317.

A plea for articles of the sort described in MR by the comment "expository paper", for continued Institutes for Teachers, and for greater clarity in research papers.

Stone, Marshall H. Mathematics and the future of science. Bull. Amer. Math. Soc. 63 (1957), 61–76.

The thirtieth Josiah Willard Gibbs lecture, delivered at Rochester, New York on December 27, 1956, under the auspices of the American Mathematical Society.

★ **Narlikar, V. V. From the trivial to the non-trivial through isomorphisms between thought and nature.** Rama Krishna Das, Banaras Hindu University Press, Banaras, 1956. 24 pp.

An address on the interrelations of mathematics and physics.

★ **Colerus, Egmont. Mathematics for everyman: From simple numbers to the calculus.** Emerson Books, Inc., New York, 1957. xi+255 pp. $3.95.

★ **Strunz, Kurt. Pädagogische Psychologie des mathematischen Denkens.** 2 Aufl., Quelle und Meyer, Heidelberg, 1956. 180 pp. (2 plates). DM 12.80.

van Hiele, P. M.; and van Hiele-Geldof, D. The educational value of mathematics. Euclides, Groningen 32 (1956/57), 277–281. (Dutch)

★ **Lass, Harry. Elements of pure and applied mathematics.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1957. xi+491 pp. $7.50.

The ten chapter headings are: linear equations, determinants and matrices; vector analysis; tensor analysis; complex-variable theory; differential equations; orthogonal polynomials, Fourier series and integrals; Stieltjes integral, Laplace transform, and calculus of variations; group theory and algebraic equations; probability theory and statistics, real-variable theory.

There are problems at the end of each section and many examples.

★ **Nicolle, Jacques. La symétrie.** Presses Universitaires de France, Paris, 1957. 119 pp.

A popular account with nine chapters: definitions élémentaires, opérations de symétrie, ensembles finis, ensembles infinis, les cristaux, biologie et symétrie, chimie et symétrie, les phénomènes physiques et la prévision, problèmes divers.

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